

# Microscopic Theory of a System of Interacting Bosons: A Unifying New Approach

YATENDRA S. JAIN

Department of Physics  
N.E.H.U., Shillong-793 022, Meghalaya, India

**Abstract**—This paper reports an entirely new approach to the microscopic understanding of the behavior of a system of interacting bosons such as liquid  ${}^4\text{He}$ . It reveals that each particle in the system represents a  $(\mathbf{q}, -\mathbf{q})$  pair (SMW pair) moving with a center of mass momentum  $\mathbf{K}$ . An energetically *weak effect*, resulting from inter-particle attraction and the overlap of wave packets, locks these particles in phase ( $\phi$ ) space at  $\Delta\phi = 2n\pi$  (with  $n = 1, 2, 3, \dots$ ) and leads them to acquire a kind of collective binding. The entire system below  $\lambda$ -point behaves like a single macroscopic molecule. The binding is identified as an energy gap between the superfluid and normal states of the system. The  $\lambda$ -transition, resulting from inter-particle quantum correlations, is the onset of an *order-disorder* of particles in their  $\phi$ -space and their *Bose Einstein condensation* (BEC) in the ground state of the system defined by  $q = \pi/d$  and  $K = 0$ . The fractional density of condensed particles ( $n_{K=0}(T)$ ) varies monotonically from  $n_{K=0}(T_\lambda) = 0$  to  $n_{K=0}(0) = 1.0$ . The  $\lambda$ -transition represents the occurrence of twin phenomena of broken gauge symmetry and phase coherence. In variance with the conventional belief, it is concluded that the system can not have  $p = 0$  condensate. In addition to the well known modes of collective motions such as phonons, rotons, maxons, *etc.*, the superfluid state also exhibits a new kind of quantum quasi-particle, *omon*, characterized by a phononlike wave of the oscillations of momentum coordinates of the particles. The theory explains the properties of *He-II*, including the origin of quantized vortices, critical velocities, logarithmic singularity of specific heat and related properties, *etc.*, at the quantitative level. It conforms to the excluded volume condition, microscopic and macroscopic uncertainty, and vindicates the two fluid theory of Landau, an idea of macroscopic wave function envisaged by London, *etc.* As discussed elsewhere in this journal, the framework of this theory can also help in unifying the physics of widely different systems of interacting bosons and fermions.

**Keywords:** bosonic system — superfluidity — microscopic theory — many body system

## 1.0: Introduction

*Liquid*  ${}^4\text{He}$  (LHE-4), a *system of interacting bosons* (SIB), has been investigated extensively for its unique behavior at *low temperature* (LT). The wealth of experimental and theoretical results has been reviewed in several books and articles, (*e.g.*, references 1–10) and other references cited therein. In the

process of cooling LHe-4 transforms from its normal (N) phase (*He-I*) to superfluid (S) phase (*He-II*) at  $T_\lambda = 2.17$  K. *He-II* exhibits several interesting properties including superfluidity, *i.e.*, zero viscosity ( $\eta = 0$ ) for its flow through narrow channels. All these properties, including the fact that helium remains liquid down to absolute zero, can not be understood as classical behavior. They are, obviously, connected with a quantum behavior at the macroscopic level and LHe-4 provides unique opportunity for its investigation, indicating that the investigations are of fundamental importance.

Soon after the experimental discovery<sup>11</sup> of superfluidity of *He-II* London<sup>12</sup> proposed that the phenomenon should arise from  $p = 0$  condensate defined by  $n_{p=0}(T)$  (a macroscopically large fraction of  $^4\text{He}$  atoms occupying a single particle state of momentum  $p = 0$  as a result of their *Bose Einstein condensation* (BEC)<sup>5</sup>). The idea was imported from a well known conclusion that a *system of non-interacting bosons* (SNIB) below certain temperature ( $T_{\text{BEC}}$ )<sup>5,13</sup> has non-zero  $n_{p=0}(T)$  increasing smoothly from  $n_{p=0}(T_\lambda) = 0$  to  $n_{p=0}(T = 0) = 1.0$  through  $n_{p=0}(T < T_\lambda) < 1.0$ . It was criticized at an initial stage by Landau<sup>14</sup> because LHe-4 is not a SNIB. However, it received grounds when Bogoliubov's theory of weakly interacting bosons<sup>15</sup> concluded that the increasing strength of interactions *simply* depletes the magnitude of  $n_{p=0}(T)$ . Widely different mathematical models such as Bogoliubov prescription<sup>15</sup>, pseudo-potential technique<sup>16</sup>, Jastrow formalism<sup>17,18</sup>, Feenberg's perturbation<sup>2,18</sup>, *etc.*, have been used to calculate  $n_{p=0}(T)$  assumed to exist in *He-II* and to develop a general framework of microscopic theories hereafter known as *conventional theories* (CTs). While recent papers by Moroni *et al.*<sup>19</sup> and Sokol<sup>20</sup> provide the present status of our conventional understanding of LHe-4, several articles published in a recent book<sup>10</sup> discuss the role of BEC in accounting for the LT behavior of widely different many body quantum systems. It appears that the CTs follow two different approaches, **A1** and **A2**.

**A1:** This approach has been developed by Bogoliubov<sup>15</sup>, Beliaev<sup>21,22</sup>, Hugenholtz and Pines<sup>23</sup>, Lee and Yang<sup>24</sup>, Brueckner and Sawada<sup>25</sup>, Wu<sup>26</sup>, Sawada<sup>27</sup>, Gavoret and Nozieres<sup>28</sup>, Hohenberg and Martin<sup>29</sup>, and DeDominicis and Martin<sup>30</sup> and reviewed by Woods and Cowley<sup>4</sup> and Toyoda<sup>31</sup>. While the approach has been introduced elegantly by Fetter and Waddecka<sup>32</sup>, its important aspects have been summed up, recently, by Nozieres<sup>33</sup> and Huang<sup>34</sup>. As such, a theoretical formulation following this approach starts with the hamiltonian written in terms of the second quantized Schrodinger field and then it proceeds by using several important inferences such as: (i) the *hard core* (HC) potential can be used perturbatively by using a method such as pseudopotential method<sup>16</sup>, (ii) coupling constant can be related directly to the two body *scattering length* ( $\sigma$ )<sup>22</sup>, (iii) a dimensionless expansion parameter ( $n\sigma^3$ ) can be used to perform perturbation calculation<sup>31</sup>, and (iv) the so called "depletion" effect which is related to  $n_{p=0}(T)$  can be treated in the conventional perturbation method by introducing *c*-number operator for  $p = 0$  bosons<sup>35</sup>. Finally, using the quantum statistical mechanical expectation value of the second quantized field

operator as an *order parameter* (OP) of the transition<sup>23,29</sup>, it calculates relevant part of *free energy* ( $F$ ) as a function of this OP in order to explain superfluidity and related properties. For this purpose one uses sophisticated mathematical tools as discussed in<sup>31,36</sup>.

**A2:** Theoretical formulations using this approach aim at finding *radial distribution function* (RDF),  $g(r)$ , and liquid structure factor,  $S(Q)$ , which can be used to calculate the *ground state* (G-state) properties, excitation spectrum, different thermodynamic properties and equation of state; the superfluid behavior is explained in terms of one body density matrix  $\rho(r)$  whose asymptotic value at large  $r$  gives the condensate fraction  $n_{p=0}(T)$ <sup>37</sup>. While different aspects of the approach and related subject have been discussed by Feenberg<sup>2</sup>, Croxton<sup>38</sup>, Ceperley and Kalos<sup>39</sup> and Yang<sup>37</sup>, its application to LHE-4 has been reviewed elegantly by Smith *et al.*<sup>40</sup>, Campbell and Pinski<sup>41</sup>, Schmidt and Pandharipande<sup>42</sup>, Reatto<sup>43</sup>, and Ristig and Lam<sup>44</sup>. Widely different methods of computer calculation of  $g(r)$ ,  $S(Q)$  and other properties of the system are discussed and reviewed in references 45–47. The approach has been used, recently, by Moroni *et al.*<sup>19</sup> and Kallio and Piilo<sup>48</sup> to determine the properties of LHE-4 and electron gas, respectively. A comprehensive list of important papers related to this approach is also available in these papers<sup>19,48</sup>.

As such, a very large number of theoretical papers have been published on the subject<sup>49</sup>. However, neither a microscopic theory that explains the properties of LHE-4 could emerge nor has the existence of non zero  $n_{p=0}(T)$  in *He-II* been proved experimentally beyond doubt. Reviewing the progress of theoretical work at the fifteenth Scottish University summer school (1974), Rickayzen<sup>50</sup> stated, “There is no microscopic theory of superfluid <sup>4</sup>He. There are many mathematical models which appear to provide insight into the behavior of superfluids but there is no theory which provides quantitative prediction that agree with observation. Even some of the most widely held assumptions of the theory such as the idea of condensation in a zero momentum state can not be said to be proved beyond reasonable doubt.” Concluding a review article, Woods and Cowley<sup>4</sup> also observe, “Despite all the experimental information and the numerous theoretical discussions there is still no convincing theory of the excitations which begins with the known interaction between helium atoms.” At the same time Kleban<sup>51</sup> proved that theories assuming the existence of  $p = 0$  condensate contradict the *excluded volume condition* (EVC)—a direct consequence of HC nature of particles; however, his work was not given due importance. In a recent article Sokol<sup>20</sup> also observes that: (i) neutron inelastic scattering experiments have so far not given any indication of the direct evidence (*i.e.*, a  $\delta$ -function peak in the momentum distribution,  $n(p)$ , at  $p = 0$ ) of the existence of  $p = 0$  condensate ( $n_{p=0}(T)$ ) in *He-II* and it is unlikely that this goal will ever be reached, and (ii) in the absence of detailed microscopic theory the estimate of  $n_{p=0}(0) \approx 0.1$  from different experiments depends on current theories, models, and empirical relations used to interpret experimental results and if their underlying assumptions are incorrect, such estimates of

$n_{p=0}(T)$  will have no meaning. Evidently, the accuracy of our present understanding about the existence of  $n_{p=0}(T)$  is seriously doubted and a correct microscopic theory of the system is still awaited.

As such, the only way we can understand the properties of LHE-4 is the two fluid phenomenology of Landau<sup>14</sup> supplemented by the idea of quantized circulation presented by Onsager<sup>52</sup> and Feynman<sup>53</sup>. However, this combination also has several shortcomings which have been discussed in detail by Putterman<sup>5</sup> who remarks, “The two fluid theory plus quantization bears striking similarity to the old quantum theory. Just as the old quantum theory had to be modified in light of the wave particle duality, one must expect that our deterministic two fluid theory of *He-II* will be modified by macroscopic wave particle duality.” Underlining the failure of numerous efforts to conclude a theory that explains the properties of *He-II*, Putterman<sup>5</sup> also speculates that perhaps the theory of natural phenomena, *the Wave Mechanics*, is not equipped with basic principles to explain the superfluidity of *He-II*. The above mentioned facts motivated us to use an entirely new approach to develop the long awaited theory of LHE-4 type SIB reported in this paper. Our approach uses wave nature and related aspects in their right perspective and, for the reasons listed in Section 2.8, envisages that the origin of superfluidity and related properties of LT phase of a SIB lies in the BEC condensation of  $(\mathbf{q}, -\mathbf{q})$  pairs in a state of their center of mass momentum  $\mathbf{K} = 0$ . This is one of the important differences in our approach and the conventional approach which emphasizes BEC of particles in a single particle state of  $p = 0$ . The salient aspects of our theory are available in Jain<sup>54</sup>.

The paper has been arranged as follows. Starting from a  $N$ -particle microscopic quantum hamiltonian and using the usual method of solving a Schrodinger equation, Section 2.0 concludes the basic form of the state functions of  $N$  HC particles and analyzes the evolution of a bosonic system with falling  $T$ , G-state energy of the system, effective inter-particle potential and  $(\mathbf{q}, -\mathbf{q})$  pair condensation as a natural basis of  $\lambda$ -transition, *etc.* While Section 3.0 determines the relations for quantum correlation potential controlling the position of particles in phase space, Section 4.0 finds the relation for  $T_\lambda$ . While different aspects of the S-phase of the system, *e.g.*, the configurations of particles, nature of thermal excitations, the origin of two fluid behavior,  $(\mathbf{q}, -\mathbf{q})$  bound pairs and energy gap between S- and N-configurations, superfluidity and related properties, quantized vortices, broken gauge symmetry or off diagonal long range order (ODLRO), logarithmic singularity of specific heat, *etc.*, are studied in Section 5.0, the thermodynamic behavior of the N-phase is analyzed in Section 6.0 and how best our theory accounts for the properties of *He-II* at quantitative scale is discussed in Section 7.0. Comparing the salient aspects of A1 and A2 approaches of CTs with corresponding aspects of our approach in Section 8.0, we make some important concluding remarks in Section 9.0 and discuss the free energy and order parameter of  $\lambda$ -transition in Appendix A. Finally, it may be mentioned that a critical analysis of the wave mechanics of

two HC particles, which serves as an important basis of this work, and the way the new approach of this work may help in unifying our microscopic understanding of widely different systems of interacting bosons and fermions are discussed elsewhere in this journal<sup>55,56</sup>.

## 2.0: Basic Aspects of Theory

### 2.1: The System

The microscopic quantum hamiltonian of a system of  $N$ -interacting bosons can be expressed as

$$H(N) = -\frac{\hbar^2}{2m} \sum_i^N \nabla_i^2 + \sum_{i < j}^N V_{ij}(r = |\mathbf{r}_i - \mathbf{r}_j|) \quad (1)$$

where notations have their usual meaning. The important aspects of the system can be summed up by the following points:

2.1.1. To a *good approximation*  $V_{ij}(r)$  could be considered as the sum of: (i) the repulsive potential  $V_{ij}^R(r)$  approximated to  $V_{\text{HC}}(r)$  (defined by  $V_{ij}(r < \sigma) = \infty$  and  $V_{ij}(r \geq \sigma) = 0$  with  $\sigma = \text{HC diameter}$ ), and (ii) a relatively long range weak attraction,  $V_{ij}^A(r)$ .

2.1.2. As  $V_{ij}^A(r)$  can be replaced by a constant negative external potential (say  $-V_0$ )<sup>18</sup>, at the first stage we are, therefore, left with only  $V_{\text{HC}}(r)$  to deal with. *As such the particles in the system represent hard balls moving freely on the surface of constant potential  $-V_0$ .* The only role of  $V_{\text{HC}}(r)$  is to restrict particles (in fact we should better say their representative wave packets) from sharing a single  $r$  point.

2.1.3. While the main role of  $V_{ij}^A(r) \approx -V_0$  is to keep all particles confined to a fixed volume ( $V$ ), its small but important effect leads the particles to have: (i) their relative phase position ( $\Delta\phi$ ) locked at  $\Delta\phi = 2n\pi$  (with  $n$  being an integer number) (*cf.* Section 3.0), and (ii) a kind of collective binding among all particles of the system as concluded in Section 5.4, where we study the second stage role of  $V_{ij}^A(r)$  by using it as a perturbation on the states of HC particles and find that the system possesses an energy gap between its N- and S-states.

2.1.4. Although translational invariance of fluids implies that momentum of a particle be a good quantum number<sup>18, p. 361</sup> and the only sensible single particle function should be a plane wave. However, since the use of plane waves does not forbid possible overlap of any two HC particles we have serious difficulties in avoiding the divergence of  $\langle H(N) \rangle$  to  $\infty$ . To this effect, while CTs use methods such as pseudopotential technique<sup>16</sup>, Jastrow formalism<sup>17,18</sup>, *etc.*, this theory uses an entirely new approach based on several new inputs summed up in the following section.

### 2.2: New Inputs

In what follows from Section 2.1, the dynamics of two HC particles should be used as an important basis of the theory of a SIB. Analysing these dynamics

at length, we discovered its several untouched aspects<sup>55</sup> which serve as the new inputs of our theory that can be summarized as follows:

2.2.1: *The correct pair waveform.* While the dynamics of two HC particles (say, P1 and P2) in a SIB can be described by the hamiltonian<sup>55</sup>

$$H(2) = \left[ -\frac{\hbar^2}{4m} \nabla_R^2 - \frac{\hbar}{m} \nabla_r^2 + A \cdot \delta(r) \right], \quad (2)$$

the pair waveform

$$U^\circ = \psi_k(r)^\circ \cdot \exp(i\mathbf{K} \cdot \mathbf{R}) \cdot \exp[-i(\varepsilon(K))], \quad (3a)$$

with

$$\psi_k(r)^\circ = \sqrt{2} \cdot \cos[(\boldsymbol{\alpha} + \mathbf{k} \cdot \mathbf{r}/2)] \cdot \exp[-i\varepsilon(k)t/\hbar] \quad (3b)$$

describing their relative motion represents an eigenfunction of  $H(2)$ . We have  $\mathbf{k} = 2\mathbf{q}$ ,  $\boldsymbol{\alpha} = 0$  for  $U^+$  (or  $\psi_k(r)^+$ ),  $\boldsymbol{\alpha} = \boldsymbol{\pi}$  for  $U^-$  (or  $\psi_k(r)^-$ ) and other notations having their usual meaning<sup>55</sup>. We also have

$$E(2) = \langle H(2) \rangle = \frac{\hbar^2 K^2}{4m} + \frac{\hbar^2 k^2}{4m} \quad (4)$$

2.2.2:  $\lambda/2 \leq d$  condition. P1 and P2 in their  $\psi_k(r)$  maintain a center of symmetry at their CM and satisfy  $\lambda/2 \leq \langle r \rangle$  condition (cf. Equations 4 and 5) and Section B(3) of reference 55). For a system like liquid  $^4\text{He}$ ,  $\lambda/2 \leq \langle r \rangle$  would read as  $\lambda/2 \leq d$ , where  $d$  is the average nearest neighbor separation decided absolutely by the inter-particle interactions. If two neighboring particles happen to have  $\lambda/2 > d$ , they would have mutual repulsion (known as zero point repulsion) forcing an increase in  $d$ . However, if the inter-particle interactions do not permit desired increase in  $d$ , the pair would absorb necessary energy from interacting surrounding to have  $\lambda/2 = d$ .

2.2.3: *Inter-particle phase correlation.*  $\psi_k(r)^\pm$ , being the superposition of two plane waves of  $\mathbf{q}$  and  $-\mathbf{q}$  momenta, represents a *stationary matter wave* (SMW) that modulates the probability

$$|\psi_k(r)^\circ|^2 = |U^\circ|^2 = 2 \cdot \cos^2[(\boldsymbol{\alpha} + \mathbf{k} \cdot \mathbf{r})/2] = g(\phi) \quad (5)$$

of finding two particles at a separation,  $\phi = \mathbf{k} \cdot \mathbf{r}$ , in  $\phi$ -space, with  $g(\phi)$  representing their inter-particle  $\phi$ -correlation. As such the pair adopts a  $(\mathbf{q}, -\mathbf{q})$  pair configuration and it is, rightly, identified<sup>55</sup> as  $(\mathbf{q}, -\mathbf{q})$  pair or SMW pair. For the first time,  $\phi$ -correlation is identified to play an important role in deciding the LT properties of the system. To what extent the radial distribution factor  $g(r)$  (i.e., the  $r$ -correlation) remains relevant is discussed in Section 5.1.

2.2.4: *Representation of a particle by  $U^-$  pair waveform.* Each particle in a SMW pair can be represented either by  $U^+$ , when presumed to be in its *self su-*

perposition (SS) state, or by  $U^-$ , if presumed to represent a *mutual superposition* (MS) state of the pair. However, since  $U^+$  and  $U^-$  are equivalent (*cf.* Section C.2 of reference 55), we propose to use

$$U^-(i) = \sqrt{2} \sin[(\mathbf{q} \cdot \mathbf{r}_i)/2] \cdot \exp(i\mathbf{K} \cdot \mathbf{R}_i) \cdot \exp[-i(\varepsilon(K) + \varepsilon(q))t/\hbar], \quad (6)$$

to describe  $i$ th ( $i = 1$  or  $2$ ) particle of the SMW pair because it embodies the fact that the pair waveform of two HC particles must vanish at their CM  $r_1 = r_2 = 0$ , more clearly than  $U^+$ . However, in doing so  $R_i$  in Equation 6 needs to be identified as the position of the CM of  $i$ th particle (not the CM of the pair) and  $r_i$  as the coordinate of a point (within the WP of  $i$ th particle) measured from a nodal point of  $\psi_k(r)$  on the line joining the two particles in a pair. The single particle energy and momentum operators for the  $K$ -motion should be  $-(\hbar^2/8m) \cdot \nabla_{R_i}^2$  and  $-i(\hbar/2) \cdot \nabla_{R_i}$ , respectively. It appears that each particle has two motions: (i) the  $q$ -motion of energy  $\varepsilon(q) = \hbar^2 q^2/2m$  and (ii) the  $K$ -motion of energy  $\varepsilon(K) = \hbar^2 K^2/8m$ .

2.2.5:  $(\mathbf{q}, -\mathbf{q})$  Pair condensation. Because each particle in the system represents a  $(\mathbf{q}, -\mathbf{q})$  pair moving with CM momentum  $K^{55}$ ,  $\lambda$ -transition should, naturally, be a consequence of BEC of such pairs. Other strong reasons for  $(\mathbf{q}, -\mathbf{q})$  pair condensation to be a basis of  $\lambda$ -transition are discussed in detail in Section 2.8.

### 2.3: $U^-$ Pair Waveform and Rearrangement of $H_0(N)$

Representation of each particle of the system by separate  $U^-$  pair waveform (as proposed above) requires rearrangement of

$$H_0(N) = H(N) - \sum_{i>j} V_{ij}(r_{ij}) \equiv H(N) - A \cdot \sum_{i>j} \delta(r_{ij}),$$

for their compatibility. We can pair particles in a cyclic order by rearranging  $H_0(N)$

$$H_0(N) = \sum_i^N h_i = \sum_i^N \frac{1}{2} [h_i + h_{i+1}] = \sum_i^N h(i), \quad (7)$$

where we assume that  $h_{N+1} = h_1$  and define

$$h_i = -\frac{\hbar^2}{2m} \nabla_i^2 \quad \text{and} \quad h(i) = -\frac{\hbar^2}{8m} \nabla_{R_i}^2 - \frac{\hbar^2}{2m} \nabla_{r_i}^2. \quad (8)$$

Alternatively we can also have

$$\begin{aligned} H_0(N) &= \sum_i^N h_i = \sum_j^{N/2} \frac{1}{2} [h_{2j-1} + h_{2j}] + \frac{1}{2} [h_{2j} + h_{2j-1}] \\ &= \sum_{j=1}^{N/2} h(2j-1) + h(2j). \end{aligned} \quad (9)$$

Apparently, Equation 7 can be preferred over Equation 9 because the latter demands  $N$  to be even. For the same reason one can also prefer another method discussed in our earlier report<sup>54</sup>, which has no such requirement, and it also helps in including all possible SS as well as MS states that we may count for any two particles in the system. However, since our basic objective is to rearrange  $H_0(N)$  as a sum of  $N$   $h(i)$  terms compatible with  $N$  separate  $U^-$  pair waveforms for  $N$  particles, one can use any suitable rearrangement. The details of the method of rearrangement are unimportant.

#### 2.4: State Functions of $N$ HC Particles

In what follows from Section 2.2.4, each particle in the system can be represented by the pair waveform  $U^-$  (Equation 6) and we can construct a state function of the system by following the standard procedure. Evidently, for  $N$  particles we have  $N$  different  $U^-$  rendering  $\Sigma = N!$  different  $\Psi_n$  state functions of  $n$ th quantum state of equal  $E_n$ . Using

$$E_n(K) = \sum_i^N \varepsilon(K)_i \quad \text{and} \quad E_n(k) = \sum_i^N \varepsilon(k)_i, \quad (10)$$

we have

$$\Psi_n = \phi_n(q) \cdot \phi_n(K),$$

$$\phi_n(q) = \left[ \left( \frac{2}{V} \right)^{N/2} \prod_{i=1}^N \sin(\mathbf{q}_i \cdot \mathbf{r}_i) \right] \exp[-i E_n(k)t/\hbar], \quad (10b)$$

$$\phi_n(K) = \left[ A \left( \frac{1}{V} \right)^{N/2} \sum_{pK} (\circ 1)^p \prod_{i=1}^N \exp[i(\mathbf{K}_i \cdot \mathbf{R}_i)] \right] \exp[-i E_n(K)t/\hbar],$$

with  $A = (1/N!)^{1/2}$ . Here  $\sum_{pK} (\pm 1)^p$  refers to the sum of different permutations of  $K$  over all particles. While the use of  $(+1)^p$  or  $(-1)^p$  in Equation 10b depends on the bosonic or fermionic nature of the system for the spin character of its particles, the use of the restriction  $q_j \geq \pi/d$  in Equation 10a treats the so called fermionic behavior (in the  $r$ -space) of HC particles<sup>18</sup>,—bosons and fermions alike. Evidently, a state function of  $N$  HC bosons should differ from that of  $N$  HC fermions in the choice of  $(+1)^p$  or  $(-1)^p$ . Note that  $\Sigma$  different  $\Psi_n$  counted above take care of the permutation of  $k = 2q$ . We have

$$\Phi_n = \frac{1}{\sqrt{\Sigma}} \sum_i^{\Sigma} \Psi_n^{(i)}, \quad (11)$$

which represents the general form of a state function that should reveal the physics of the system. Note that  $\Phi_n$  represents a state where each particle, as a WP of size  $\pi/q$ , has a plane wave motion of momentum  $\mathbf{K}$ .



### 2.5: G-State Energy

Following our other paper<sup>55</sup>, which concludes that for any two particles represented by  $U^-$  pair waveform, we have  $\langle A \cdot \delta(r_{ij}) \rangle = 0$ , which gives

$$\begin{aligned}
 E_n &= \frac{\langle \Phi_n | H_0(N) + A\delta(r_{ij}) | \Phi_n \rangle}{\langle \Phi_n | \Phi_n \rangle} = \frac{\langle \Phi_n | \left\{ \sum_i^N h(i) \right\} | \Phi_n \rangle}{\langle \Phi_n | \Phi_n \rangle} \\
 &= \sum_i^N \left( \frac{\hbar^2 K_i^2}{8m} + \frac{\hbar^2 q_i^2}{2m} \right), \quad (12)
 \end{aligned}$$

where each particle is presumed to represent a SMW pair of CM momentum  $\mathbf{K}_i$  and relative momentum  $\mathbf{k}_i = 2\mathbf{q}_i$ . While  $K_i$  can have any value ranging between 0 to 1, the lowest  $q_i$  is decided by the volume  $v_i$  of the cavity (formed by neighboring particles) exclusively occupied by it. As such, in the G-state of the system we have all  $K_i = 0$ . To fix the possible value(s) of  $q_i$  for which  $E_0$  has its minimum value, we note that the condition  $\lambda/2 \leq d$  implies that a spherical volume of diameter  $\lambda/2$  belongs exclusively to a HC particle of  $\lambda/2 > \sigma$  and each particle in the G-state has lowest possible energy, *i.e.*, largest possible  $\lambda/2$ , the net G-state energy of the system should be

$$E_0 = \sum_i^N \frac{h^2}{8m v_i^{2/3}} \quad \text{and} \quad \sum_i^N v_i = V \quad (\text{constant}) \quad (13)$$

if particles are assumed to occupy different  $v_i$ . Simple algebra reveals that  $E_0$  has its minimum value for  $v_1 = v_2 = \dots v_N = V/N$ . Obviously,

$$E_0 = N h^2 / 8 m d^2 = N \mathcal{E}_0. \quad (14)$$

Note that inter-particle interactions enter in deciding the G-state energy through  $d$ . In sharp contrast with  $E_0$  obtained from CTs<sup>1,18</sup> our  $E_0$  does not depend on  $\sigma$ . The accuracy of this aspect of our result is well evident because two particles of  $\lambda/2 > \sigma$  cannot resolve the HC structure within the larger size WPs of each other. The belief in such possibility would contradict the basic principle of image resolution.

### 2.6: Evolution of the System with Decreasing $T$

For a constant particle density of the system,  $(d - \lambda/2)$  decreases with decreasing  $T$ . In the process at certain  $T = T_c$ , when  $d - \lambda/2$  vanishes at large,  $q$  motions get freezed into zero point motions of  $q = q_0 = \pi/d$ . Evidently, the system moves from a state of  $\lambda/2 \leq d$  to that of  $\lambda/2 = d$ . While the former state of  $\phi (= 2qd) \geq 2\pi$  represents randomness of  $\phi$  positions and, therefore, a disorder in  $\phi$ -space, the latter state of  $\phi = 2\pi$  defines an ordered state. Thus the particles in the system move from their disordered to ordered state in  $\phi$ -space at  $T_c$ . With all  $q_j = q_0$ , different  $\Psi_n^{(i)}$  of Equation 11 become identical and  $\Phi_n$  attains the form of a single  $\Psi_n$  (*cf.* Equation 10). As such, all the  $\Sigma$  microstates merge

into one at  $T_c$ , indicating that the entire system attains a kind of oneness<sup>57</sup>; the system at  $T \leq T_c$  is, therefore, described by

$$\Phi_n(S) = \phi_n^0(q_0) \cdot \phi_n^e(K), \quad (15)$$

which is obtained by replacing all  $\mathbf{q}_j \cdot \mathbf{r}_j$  in Equation 10 by  $q_0 r$  as for a given  $r$  and  $\mathbf{q} \cdot \mathbf{r} = \pi$  (i.e.,  $r \cos \theta = \lambda/2$ ), lowest energy (or largest  $\lambda$ ) configuration demands  $\theta = 0$ ; as such, each particle seems to be in s-wave state.

### 2.7: Effective Inter-Particle Interaction

In order to discuss the level of accuracy, to which our theory accounts for the real interaction,  $V_{ij}(r)$ , we note that our theory not only replaces  $V_{ij}^R(r)$ , as an approximation, by  $V_{\text{HC}}(r) \equiv A\tilde{\alpha}(r)$  but also imposes a condition, “that two WPs of HC particles should not share any point  $\mathbf{r}$  in configurational space”, —equivalent to assuming the presence of a repulsion of *finite range*,  $r_a = \lambda/2$ ; it appears that the WP manifestation of particle extends the range of the influence of  $V_{\text{HC}}(r)$  from  $r_a = \sigma$  to  $r_a = \lambda/2$  when  $\lambda/2 > \sigma$ . This repulsion is nothing but the zero point repulsion<sup>58</sup> which can be derived as the first  $d$  derivative of  $\epsilon_0 = h^2/8md^2$  representing the G-state energy of a particle. Evidently,  $A$  in  $V_{\text{HC}}(r) \equiv A\tilde{\alpha}(r)$  should be proportional to  $h^2/8md^2$ .

Further, since  $V_{ij}^R(r)$  in most SIB falls faster (in LHE-4 it varies as  $r^{-12}$ ) than the zero-point repulsion, varying as  $r^{-2}$ , the latter would dominate  $V_{ij}^R(r)$  particularly for all  $r > \sigma$  and  $\leq \lambda/2$  and this observation agrees with the experimental facts that: (i) LHE-4 does not solidify due to the operation of zero-point repulsion even at  $T = 0$  unless an external pressure of  $\approx 25$  atm is applied, and (ii) it exhibits volume expansion with falling  $T$  around  $T_\lambda$ <sup>1</sup>. We also note that our condition  $\lambda/2 \leq d$  identifies  $d$  as the upper limit of the WP size  $\lambda/2$  (the key aspect of our theory) and  $d$  is decided by  $V_{ij}(r)$  without any approximation. These observations, clearly, prove that *our theory accounts for the  $V_{ij}^R(r)$  and  $V_{ij}^A(r)$  components of  $V_{ij}(r)$  close to their real effect.*

### 2.8: Why $(\mathbf{q}, -\mathbf{q})$ Pair Condensation

The phenomenon of superfluidity/superconductivity of a fermionic system is attributed to the condensation of Cooper pairs of fermions for a reason that the Pauli exclusion principle forbids two identical fermions from occupying single energy state, while any number of these pairs presumed to behave like bosons can do so. Because Pauli exclusion does not apply to bosons, conventional theorists found no difficulty in assuming the condensation of macroscopically large number of bosons into a single particle state of  $p = 0$  as their main theme. However, this assumption ignores the fact that in the same way that two fermions do not occupy same point in  $k$ -space, two HC particles do not occupy the same point in  $r$ -space. This is particularly important because the requirement of antisymmetry of two fermion wave function,  $\Phi_a(1, 2) =$

$[v_{\mathbf{k}}(\mathbf{r}_1) \cdot v_{\mathbf{k}'}(\mathbf{r}_2) - v_{\mathbf{k}'}(\mathbf{r}_2) \cdot v_{\mathbf{k}}(\mathbf{r}_1)]$  for their exchange, makes  $\Phi_a(1, 2)$  vanish not only for  $\mathbf{k}' = \mathbf{k}''$  but also for  $\mathbf{r}_1 = \mathbf{r}_2$ . Evidently, if  $\Phi(1, 2)$  of two HC particles is subjected to a condition that it should vanish for  $\mathbf{r}_1 = \mathbf{r}_2$ ,  $\Phi(1, 2)$  has to be identically antisymmetric and would, obviously, vanish also for  $\mathbf{k}' = \mathbf{k}''$ . This implies that two HC quantum particles in  $r$ -space behave like two fermions behave in  $k$ -space and concludes that two HC particles (*excluded to have*  $\mathbf{r}_1 = \mathbf{r}_2$ ) can not have  $\mathbf{k}' = \mathbf{k}''$ , particularly, in a state of their *wavemechanical superposition* (*i.e.*, a quantum state of  $\lambda > d$ ). Note that the inference would be valid not only for particles of  $\sigma \approx 0$  (*i.e.*, particles interacting through  $\delta$ -function repulsion) but also for  ${}^4\text{He}$  type atoms because finite size HC repulsion becomes equivalent to  $\delta$ -function repulsion for particles of  $\lambda/2 > \sigma^{55}$ . However, we also note that there is a difference in fermi behavior due to HC nature and that due to half integer spin; while the former excludes every particle from having  $q < \pi/d$  (applies identically to HC bosons and HC fermions), the latter excludes two particles (applies to fermions only) from having equal  $\mathbf{K}$  (Equation 10). Evidently, this excludes the possibility of nonzero  $n_{p=0}(T)$  which has also been shown to be *inconsistent with excluded volume condition of HC particles*<sup>51</sup>.

As such, like Pauli exclusion provides effective repulsion to keep two fermions apart<sup>4</sup>, the volume exclusion condition applicable to HC quantum particles and WP manifestation of quantum particles render such repulsion to keep their WPs at  $r \geq \lambda/2$ ; experimentally observed volume expansion of LHE-4 with decreasing  $T$  near  $T_\lambda^{-1}$  corroborates this fact. Evidently, there is no doubt that superfluidity of LHE-4 type SIB originates from the condensation of  $(\mathbf{q}, -\mathbf{q})$  pairs. The *binding between two particles originates from their inherent interatomic attraction* and this has been discussed in detail in Section 5.4.

### 3.0: Quantum Correlation Potential

The inter-particle *quantum correlation potential* (QCP) originating from the wave nature of particles can be obtained<sup>59</sup> by comparing the partition function (under the quantum limits of the system),  $Z_q = \sum_n \exp(-E_n/k_B T) \cdot |\Phi_n(S)|^2$  and its classical equivalent,  $Z_c = \sum_n \exp(-E_n/k_B T) \cdot \exp(-U_n/k_B T)$ . Here  $\Phi_n(S)$  is given by Equation 15. The procedure is justified because our theory describes the system by symmetrized plane waves and our assumption that only one particle occupies a single AR of these SMWs screens out the HC potential. Simplifying  $U_n$ , one easily finds that pairwise QCP has two components. The  $U_{ij}^s$  pertaining to  $k$  motion controls the  $\phi = kr$  position of a particle and we have

$$U_{ij}^s = -k_B T_0 \ln[2 \sin^2(\phi/2)], \quad (16)$$

where  $T$  has been replaced by  $T_0$  because  $T$  equivalent of  $k$  motion energy at all  $T \leq T_\lambda$  is  $T_0$ .

$U_{ij}^s$  has its minimum value  $(-k_B T_0 \ln 2)$  at  $\phi = (2n + 1)\pi$  and maximum value  $(=\infty)$  at  $\phi = 2n\pi$  occurring periodically at  $\Delta\phi = 2n\pi$  (with  $n = 1, 2, 3, \dots$ ). Since  $U_{ij}^s$

always increases for any small change  $\delta\phi$  in  $\phi$  at its minimum value, as

$$\frac{1}{2}C(\delta\phi)^2 = \frac{1}{4}k_B T_0(\delta\phi)^2 \quad \text{with force constant } C = \frac{1}{2}k_B T_0, \quad (17)$$

and particles experience a force  $= -C\delta\phi$  which tries to maintain  $\delta\phi = 0$  and the order of particles in  $\phi$ -space is sustained. Since  $U_{ij}^S$  is not a real interaction, in general, it cannot manipulate  $d$ . The  $\phi$ -space order is, therefore, achieved by driving all  $q$  towards  $q_0$ . In the S-phase, however, the interdependence of  $r$  and  $q$  through  $\phi = 2qr = 2n\pi$  makes  $V_{ij}(r)$  depend also on  $\phi$  and  $q$  positions of particles and this renders (i) a new type of inter-particle correlation known as quantum correlation (or  $\phi$ -correlation defined by  $g(\phi)$ , cf. Equation 5) which exists only in S-phase and (ii) a kind of additional potential energy (self energy, cf. Section 5.4.4) that depends on  $q$ . As such we find that unique properties of the S-phase are due to quantum correlations ( $g(\phi)$ ) rather than the normal inter-particle correlations ( $g(r)$ ) (existing in both N- and S-phases) originating from  $V_{ij}(r)$ .

The second component pertaining to  $K$  motion is expressed by

$$U_{ij} = -k_B T \ln[1 + \exp(-2\pi|R' - R''|^2/\lambda_T^2)], \quad (18)$$

with  $\lambda_T' = h/(2\pi(4m)k_B T)^{1/2}$  which is, obviously, identical to the expression obtained for non-interacting bosons<sup>59,60</sup>.  $U_{ij}$  may be seen as the origin of the force that facilitates BEC in the state of  $K = 0$  by driving particles in  $K$  space towards  $K = 0$  where it has its minimum value ( $-k_B T \cdot \ln 2$ ).

#### 4.0: The Transition

What follows from Section 2.6, the lower bound of  $T_c$  is  $T_0$  (the  $T$  equivalent of  $\mathcal{E}_0$  or that of  $\lambda = 2d$ ). This gives  $T_c = T_0 = h^2/8\pi m k_B d^2$  by using  $\lambda_T = h/(2\pi m k_B T)^{1/2}$ . To reveal the real  $T_0 = T_\lambda$ , we note that with  $T$  moving below  $T_\lambda$  particles not only have  $q = q_0$ , but also start attaining  $K = 0$ . Evidently, the  $\lambda$ -transition follows two processes simultaneously: (i) an order-disorder of particles in  $\phi$ -space rendering  $q = q_0$  and (ii) the BEC of particles (as SMW pairs) driving them towards  $K = 0$ . In view of these aspects it is evident that  $\lambda$ -transition is a second order transition. This also gives

$$T_\lambda = T_0 + \frac{1}{4}T_{\text{BEC}} = \frac{h^2}{8\pi m k_B} \left[ \frac{1}{d^2} + \left( \frac{N}{2.61V} \right)^{2/3} \right], \quad (19a)$$

where  $T_{\text{BEC}}$  is usual BEC temperature<sup>5</sup>. The factor of 1/4 appears because the plane wave  $K$  motion of a particle has  $\hbar^2 K^2/2(4m)$  energy and  $T_{\text{BEC}}$  varies as  $1/m$ . The sharpness of the transition is well evident from its condition  $\lambda = 2d$ . We analyze the nature of transition again in Appendix A by identifying its OP and the relevant part of the free energy  $F$ .

While particles in our system can be assumed, to a good approximation, to represent HC particles moving freely on a surface of constant  $-V_0$ , the fact that

two neighboring particles do experience inter-particle interaction during their relative motion needs cognizance for explaining certain experimental observations such as pressure ( $P$ ) dependence of  $T_\lambda$ . Since a relation obtained from a model based on free particle picture can be modified for the role of inter-particle interactions by replacing  $m$  by  $m^*$ , defining the effective mass of a particle, we can obtain more accurate value of  $T_\lambda$  from,

$$T_\lambda = \frac{h^2}{8\pi m^* k_B} \left[ \frac{1}{d^2} + \left( \frac{N}{2.61V} \right)^{2/3} \right], \quad (19b)$$

Evidently,  $d$ ,  $V$  and  $m^*$  are three quantities which may change with increasing  $P$ . While  $T_\lambda$  is expected to increase with increasing  $P$  for the usual decrease in the values of  $d$  and  $V$ , however,  $T_\lambda$  may show a reverse change if  $m^*$  increases with  $P$ . In this context we note that  $m^*$  for  ${}^4\text{He}$  atoms in LHE-4 should increase with  $P$  for an obvious increase in the strength of inter-particle *attraction* with increasing  $P$ . Evidently, Equation 19b can explain the  $P$  dependence of  $T_\lambda$  of LHE-4.

## 5.0: Properties of S-Phase

### 5.1: The Method of Derivation

It may be mentioned that the usual method of finding different properties of a normal fluid by using their relations with radial distribution factor  $g(r)$  can not be used to derive the properties of the LT phase clearly because this phase is dominated by  $\phi$ -correlations  $g(\phi)$ . We note that  $g(\phi)$  represents momentum correlations ( $g(q)$ ) when  $r$ -space configuration remains fixed or simply  $r$ -correlations ( $g(r)$ ) when all particles keep fixed  $q$  because the system maintains  $\Delta\phi = 2qr = 2n\pi$ . The use of only  $g(r)$  in deriving the properties of the system would, obviously, ignore  $q$ -correlations.

In view of this inference, we determined the energy of  $n$ th quantum state  $E_n = \langle H(N) \rangle$  (cf. Equation 12) by using the standard method of finding the expectation value of a physical operator. We also determined the G-state energy  $E_0$  (cf. Equation 14) by usual method of energy minimisation. Within the approximation that particles of the system move as hard balls on a sheet of constant negative potential ( $-V_0$ ) (cf. Section 2.1) and  $V_{\text{HC}}(r) \equiv A\delta(r)$ , Equation 14 renders very accurate  $E_0$ . However,  $E_n$  (Equation 12) representing higher energy state of  $K \neq 0$ , needs to be reorganized for the inter-particle phase correlations leading to collective motions of particles identified by wave vector  $Q$ . To find the energy dispersion ( $E(Q)$ ) relation for these motions and related properties, we propose to use quantum correlation potential (QCP),  $U_{ij}^s$  (Equation 16, Section 3.0) because *it expresses exactly the modulation of the particle positions by  $\Psi_n$  in  $r$ - and  $\phi$ -spaces*. In this context we are guided by the interpretation and significance of QCP as concluded by Uhlenbeck and Gropper<sup>60</sup>. Accordingly, QCP provides an alternative way of expressing inter-particle

correlations and for this purpose one has to simply introduce this potential and treat the particles classically. As such QCP provides an alternative method of plugging in our wave function  $\Phi_n(S)$  (Equation 15) with different properties of the system. The fact that QCP also provides important information such as per particle phase correlation energy ( $= -k_b T_0 \ln 2$ ), the inter-particle force constant (*cf.* Equation 17), *etc.*, which other methods can not provide, is an additional advantage of its use. As such we derive all important properties of the system in three steps.

- (i) In the first step we use QCP to find the  $\phi$ ,  $q$ , and r-space configurations of the G-state of the system defined by

$$\Delta\phi = 2n\pi, \quad q_0 = \pi/d, \quad \langle r \rangle = d \quad (20)$$

that serve as our key results followed from<sup>55</sup> and Section 3.0. Evidently, the system is a close packed arrangement of WPs. The packing leaves no freedom for particles to move across each other and they move in the order of their locations maintaining  $\Delta\phi = 2n\pi$ . We use the configuration Equation 20 to find the  $E(Q)$  in a manner, one determines excitation spectrum of crystals. In this context we also give due importance to possible changes in the configuration of the system under the influence of an excitation as well as in the nature of excited state motions at different  $Q$ .

- (ii) In the second step we recall the inter-particle attraction which was replaced, to start with, by a constant negative potential  $-V_0$  and use it as a perturbation on the SMW configuration of HC particles. This renders what we call collective binding of all particles or the energy gap between S and N states of the system, which explains superfluidity and related properties (*cf.* Section 5.5–5.9).

- (iii) Finally we use our results, such as  $\Delta\phi = 2n\pi$ , the state function (Equations 10 and 11), and the details of  $U_{ij}^s$  to find relations for quantized circulation, ODLRO, and logarithmic singularity of specific heat and related properties.

## 5.2: Thermal Excitations

*5.2.1: Approach using orderly arrangement of atoms.* From the G-state configuration (*cf.* Equation 20), we note that the system represents a close packed arrangement of WPs where particles are restored by  $U_{ij}^s$  at  $\Delta\phi = kr = 2n\pi$ , we can visualize waves of  $\phi$ -oscillations. To reveal their frequency dispersion,  $\omega_\phi(Q)$ , we consider a linear chain of atoms and only nearest neighbor interactions as the responsible forces. We have

$$\omega_\phi(Q) = \sqrt{(4C)/\beta} |\sin(Qd/2)|, \quad (21)$$

where  $Q$  is the wave vector and  $\beta$  is the measure of inertia for  $\phi$  motion. However,  $\phi$ -oscillations can appear as the oscillations of  $r$  and  $q$  because  $\delta\phi = 2q\delta r + 2\delta q r$ . We have phonons when  $q = q_0$ , and omons (a new kind of quantum quasi-particle representing a phononlike wave of the oscillations of momentum) when  $r = d$ . We note that a system like liquid  ${}^4\text{He}$  is expected to exhibit (i) no transverse mode because the shear forces between particles are negligibly small, and (ii) only one branch of longitudinal mode because the system is isotropic. Evidently,  $\omega_r(Q)$  of phonons can be represented, to a good approximation, by the dispersion of the elastic waves in a chain of identical atoms and it can be obtained from Equation 21 by replacing  $\beta$  and  $C$  by  $m$  and  $C$ , respectively. We have

$$C = 4\pi^2 C / d^2 = 2\pi^2 k_B T_0 / d^2 = \pi h^2 / 4m d^4. \quad (22)$$

However, for better accuracy  $d$  and  $C$  should, respectively, be considered descending and ascending functions of  $Q$  because increase in the energy of particles affected by an excitation reduces WP size and this renders a decrease in  $d$  and an increase in  $C$ . As such the phonon energy,  $E_{\text{ph}}(Q)$ , can be obtained from

$$E_{\text{ph}}(Q) = \hbar \omega_r(Q) = \hbar \sqrt{4C(Q)/m} |\sin(Qd(Q)/2)|. \quad (23)$$

The accuracy of this relation is also evident from the fact that the  $Q$  dependence of  $C$  and  $d$  not only explains the experimentally observed  $E_{\text{ph}}(Q)$  of  $\text{He-II}$  but also accounts for its anomalous nature at low  $Q$  (*cf.* Section 7.0). This aspect has been studied in great detail in<sup>61,62</sup>. However, since  $d \not\propto \sigma$ ,  $d(Q)$  and  $C(Q)$  are bound to become  $Q$  independent for  $Q > \pi/\sigma$  and the maximum in  $E_{\text{ph}}(Q)$  (*i.e.*, the position of so called maxon) should fall at  $Q_{\text{max}} = \pi/\sigma$  and  $E_{\text{ph}}(Q)$  over the range  $Q > \pi/\sigma$  and  $< 2\pi/d$  should follow

$$E_{\text{ph}}(Q) = \hbar \omega_r(Q) = \hbar \sqrt{4C(Q_{\text{max}})/m} |\sin(Q\sigma/2)|. \quad (24)$$

We note that phononlike dispersion can be expected till the excitation wavelength  $\Lambda > d$  (*i.e.*,  $Q < 2\pi/d$ ). The spectrum for  $\Lambda < \sigma$  is expected to follow

$$E_{\text{sp}}(Q) = \hbar^2 Q^2 / 2m_F, \quad (25)$$

a kind of *single particle* dispersion because the momentum and energy of the excitation would be carried by a single particle only; here  $m_F$  represents a kind of *mass factor that measures the effect of quantum correlation in the* ( $\mathbf{q}$ ,  $-\mathbf{q}$ ) pair.

The transition of  $E(Q)$  from  $E_{\text{ph}}(Q)$  to  $E_{\text{sp}}(Q)$  would, obviously, take place over the range,  $2\pi/d < Q < 2\pi/\sigma$ , before the  $E_{\text{ph}}(Q)$  (Equation 24) meets its zero value at  $2\pi/\sigma$ . This implies that  $E(Q)$  has to have its minimum (*the so called*

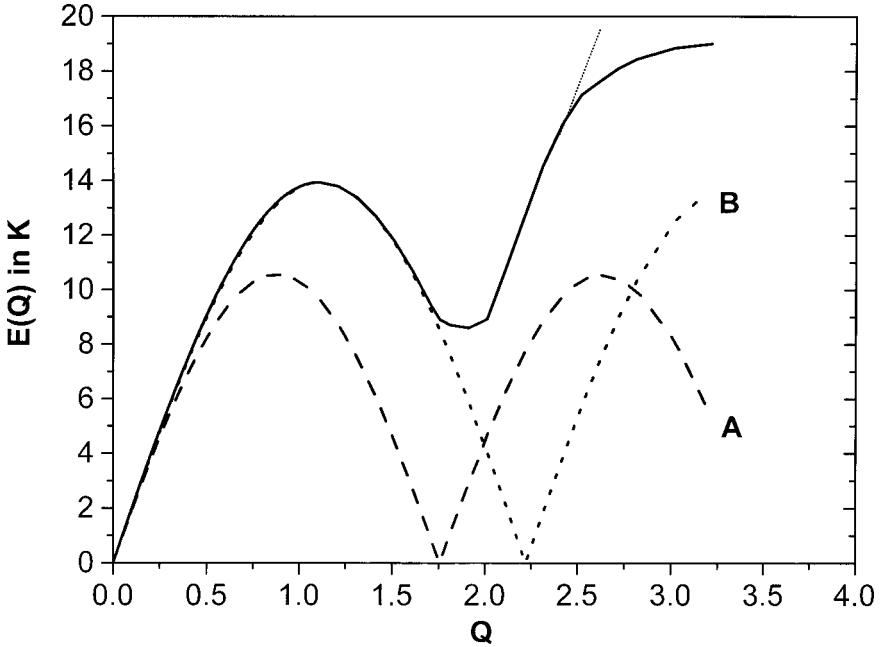


Fig. 1. Phonon dispersions of linear chains A (Curve-A) and B (Curve-B) of  ${}^4\text{He}$  atoms in a close packed arrangement of their WPs in  $\text{He-II}$  and theoretical  $E(Q)$  (Solid Line Curve) of  $\text{He-II}$  obtained by using Equations 23–27. The dotted curve beyond  $E(Q) \geq 18$  K shows the direction of  $E(Q)$  if the resonance interaction with multiphonon modes were absent. For details see Sections 5.2 and 7.0.

roton minimum) at a  $Q = Q_{\min}$  near the mid-point of  $Q = 2\pi/d$  and  $Q = 2\pi/\sigma$ . To a good approximation this gives

$$Q_{\min} \approx \left[ \frac{\pi}{d} + \frac{\pi}{\sigma} \right]. \quad (26)$$

This inference can be easily understood from Figure 1 where we depict phonon dispersion of two chains of  ${}^4\text{He}$  atoms: Chain-A is defined by largest possible  $d = 3.5787\text{\AA}$  obtained from density ( $= 0.145$  gm/cc) given in reference 1 and smallest possible  $C = 3.1681$  dyne/cm obtained from Equation 22, while Chain-B defined by shortest possible  $d$  (i.e.,  $\sigma = 2.8303\text{\AA}$ ) fixed from observed maxon position  $Q_{\max}$  ( $= \pi/\sigma = 1.11\text{\AA}^{-1}$ ) and highest possible  $C = 5.53$  dyne/cm fixed for obtaining experimentally observed maxon energy ( $= 13.92$  K) for  $\text{He-II}$  through Equation 24. It appears (cf. Figure 1) that the conditions of the system in the roton region,  $2\pi/d < Q < 2\pi/\sigma$ , are such that we simultaneously have a phonon of +ve group velocity ( $v_g$ ) on Curve-A as well as a phonon of  $-v_g$  on Curve-B. In other words, the roton is the superposition of two phonons of opposite  $v_g$  coexisting in the system. This picture of rotons resembles greatly that



suggested by Feynman<sup>63</sup>. Further, since the  $Q_{\min}$  falls close to the cross point of Curve-A and Curve-B, we have

$$E_{\text{rot}}(Q') \approx E_{\text{ph}}(Q')|_{\text{curve-A}} + E_{\text{ph}}(Q')|_{\text{curve-B}} \quad (27a)$$

or

$$E_{\text{rot}}(Q') \approx E_{\text{ph}}(Q' - 2\pi/d)|_{\text{curve-A}} + E_{\text{ph}}(Q')|_{\text{curve-B}} \quad (27b)$$

and

$$E_{\text{rot}}(Q_{\min}) \approx 2E_{\text{ph}}(Q_{\min} - 2\pi/d), \quad (27c)$$

where we have  $Q' = Q (> 2\pi/d)$ . To obtain Equations 27b and 27c we use the facts that (i)  $E_{\text{ph}}(Q')|_{\text{curve-A}} = E_{\text{ph}}(Q' - 2\pi/d)|_{\text{curve-A}}$  and (ii) the real dispersion of phonons of  $Q' - 2\pi/d$  neither follows curve-A nor Curve-B but the  $E_{\text{ph}}(Q' - 2\pi/d)$  curve, slowly drifting away from Curve-A to Curve-B (*cf.* Figure 1).

Evidently, Equations 23–27 obtained for the S-phase represent a Landau type spectrum (for example as shown for *He-II* by the solid curve in Figure 1). Using  $d(Q)$ ,  $C(Q)$  and  $m_F$  as adjustable parameters we can easily obtain a spectrum matching with experiment for *He-II* (*cf.* Section 7.0). Using Equation 22 in Equation 23, we also have

$$v_p = v_g = \sqrt{\pi}h/2md, \quad (28)$$

for low  $Q$  modes; here  $v_p$  represents phase velocity of phonons.

Finally, we note that the equation of motion of  $r_s$  (the  $r$  of  $s$ th atom), *i.e.*,

$$\partial_t^2 r_s = -\frac{1}{4}\omega_0^2[2r_s - r_{s-1} - r_{s+1}],$$

transforms into a similar equation for  $p_s = \hbar q_s$  by operating  $m\partial_t$ . This reveals  $\omega_s(Q) = \omega_q(Q)$  (the omon dispersion frequency). As concluded in Section 5.4.4, an omon is an anti-phonon quantum quasi-particle.

*5.2.2.: Feynman's approach.* Defining an excited state of the system by  $\psi = \sum_i f(r_i)\phi$  and the G-state by  $\phi$ , Feynman<sup>53,63</sup> showed that the excited state energy is minimum for  $f(r_i) = \exp(i\mathbf{k} \cdot \mathbf{r}_i)$ . He obtained

$$E(Q)_{\text{Feyn}} = \frac{\hbar^2 Q^2}{2mS(Q)}, \quad (29)$$

with  $S(Q) =$  structure factor of the system. However, for *He-II*, this relation renders an  $E(Q)$  which is found to be about two times the experimental value. Introducing back flow effects Feynman and Cohen<sup>64</sup> later obtained results of better agreement with experiments but with considerable discrepancy at high- $Q$ . In this section we apply Feynman's approach to our  $(\mathbf{q}, -\mathbf{q})$  configuration

of particles. In this context we note that under the impact of an excitation a SMW pair in the G-state configuration of  $(\mathbf{q}_0, -\mathbf{q}_0)$  and  $K = 0$ , expressed by  $U^-(\mathbf{q}_0) = U^-$  (Equation 6, with  $k = 2q_0$  and  $K = 0$ ) moves to a new configuration  $(\mathbf{q}_0 + \Delta\mathbf{q}, -\mathbf{q}_0 + \Delta\mathbf{q})$  described by

$$U^-(q_0, \Delta q) = U^-(q_0) \exp(i\mathbf{Q} \cdot \mathbf{R}) \exp(-i\varepsilon(Q)t/\hbar), \quad (30)$$

with  $2\Delta\mathbf{q} = \mathbf{Q}$ , which means that the impulse changes in the CM momentum and energy of both particles and its impact should be identified by  $\mathbf{Q}$  not by  $\Delta\mathbf{q}$ . Equation 30 further shows that  $f(R) = \exp(i\mathbf{Q} \cdot \mathbf{R})$  is the real form of  $f$  which renders  $\psi = \sum_i \exp(i\mathbf{Q} \cdot \mathbf{R}_i)\phi$ . Using these facts and recasting the relation,  $-(\hbar^2/2m)\nabla^2 f(R) = \varepsilon \int g(R - R')f(R')d^3R'$  (Equation 11.25 of Feynman<sup>63</sup>, p. 329–330), for  $i$ th particle we find  $-(\hbar^2/8m)\nabla_i^2 f(R) = \varepsilon_i(Q)S(Q)f(R)$ ; note that our single particle energy operator is  $-(\hbar^2/8m)\nabla_i^2$ . This renders  $(\hbar^2 Q^2/8m) = \varepsilon_i(Q)S(Q)$ . Adding this relation to a similar relation for  $j$ th particle of the pair gives

$$E(Q) = \varepsilon_i(Q) + \varepsilon_j(Q) = \frac{\hbar^2 Q^2}{4mS(Q)} = \frac{1}{2}E(Q)_{\text{Feyn}}, \quad (31)$$

which naturally explains  $E(Q)_{\text{expt}}$  of *He-II*. While Equation 31, as a single relation, can give us the full  $E(Q)$ , our five equations (23–27) provide a better understanding of the microscopic details of the motions responsible for its different parts.

Since Equations 23–27 have been derived by using the condition that particles in the G-state of the system are orderly placed in configurational as well as in phase space, these may appear to be inapplicable to account for the  $E(Q)$  of the N-phase, where particles have random distribution in phase space. However, in view of the discussion of Section 6.0 dealing with the thermodynamics of the N-phase, the system seems to represent nearly a close packed arrangement of WPs but without collective binding (*cf.* Section 5.4) and phase coherence. Consequently, the excitation spectra of S- and N-phases are not expected to differ significantly, particularly, at low  $Q$  and  $T$  near  $T_\lambda$ ; of course, in the absence of the phase correlation among the particles in N-phase, the energy width ( $\Delta E(Q)$ ) of excitations should increase significantly. This is found to agree with the experiments on *He-I*. Further, since the experimental  $S(Q)$  contains the information about the configurational arrangement of particles in the system, Equation 31 should provide  $E(Q)$  of its both phases with equal degree of accuracy.

### 5.3: Two Fluid Behavior

Since  $\phi_N^0(q_0)$  and  $\phi_N^e(K)$  (*cf.* Equation 15) deal separately with  $q$  and  $K$  motions of particles they represent two different components in the system. The component  $\phi_N^0(q_0)$ , representing particles in their G-state, obviously has zero entropy ( $S = 0$ ); this also has  $\eta = 0$  because particles are constrained to move in

the order of their locations (*cf.*, Section 5.1). The excitations are the effects that can propagate from one end of the system to the other against the closely packed WPs in the background. They, obviously, form a kind of gas (as envisaged by Landau<sup>14</sup>) that accounts for the total S and other thermal properties of the system. They also render a  $\eta \neq 0$  because their effects can lead to frictional movement of particles. As such,  $\phi_N^0(q_0)$  has the basic properties of S-fluid, while  $\phi_N^e(K)$  has those of N-fluid. Interestingly since all particles participate in  $\phi_N^0(q_0)$  as well as  $\phi_N^e(K)$ , none of them can be labeled as N or S particles. All these aspects justify the use of inertial mass density associated with excitations to obtain normal fluid density  $\rho_n(T)$  and superfluid density  $\rho_s(T) = \rho(T) - \rho_n(T)$  through well known relations (*cf.* p. 137 of reference 1) and vindicate two fluid theory of Landau<sup>14</sup>. It may, finally, be noted that this theory also provides new relations for obtaining  $\rho_s(T)$  and  $\rho_n(T)$  (*cf.* Section 5.5.4).

#### 5.4: Energy Gap and Self Energy

*5.4.1: Perturbative effect of attraction on SMW pair.* With  $T$  moving below  $T_\lambda$ , the WPs of neighboring particles having increased size ( $>d$ ) tend to mutually overlap and push the particles against inter-particle attraction. This overlap and  $V^A(r_{ij})$  attraction combine to perturb ( $K = 0, q_0$ ) state and its energy  $\epsilon_0$ . The attraction being a function of  $r$  can not affect CM motion. However, it can perturb the relative motion of the SMW pair. The energy of perturbed states of the pair can be determined by diagonalising the ( $2 \cdot 2$ ) energy matrix defined by  $E_{11} = E_{22} = \epsilon_0$  and  $E_{12} = E_{21} = v$  where  $\epsilon_0$  is  $q$  motion energy of each particle in the pair and  $v$  is the expectation value of the attraction. Note that particles in the pair have equal  $q$  ( $=q_0$ ). Diagonalisation of the matrix renders two states of energy  $\epsilon_0 \pm |v|$ .  $|v|$  could better be replaced by  $lv(T)$  as the overlap may depend on  $T$ . The state of lower energy ( $\epsilon_0 - lv(T)$ ) can be considered to represent a kind of bonding (or paired) state while that of  $\epsilon_0 + lv(T)$  an antibonding (or unpaired) state. This follows the well established standard method (*Molecular Orbital Theory*<sup>65</sup>) applied to a similar case in which two identical atomic orbitals form two molecular orbitals of bonding and anti-bonding nature. The pair is expected to be in bonding state provided the two WPs continue to have their overlap.

*5.4.2: Perturbation effect on a state of N-particle.* Starting from the  $N$ -body microscopic quantum hamiltonian,  $H(N)$  (Equation 1) and following the discussion of Section 5.3, we find that the states of N- and S-components of the fluid are, respectively, described by

$$H(R) = -\frac{\hbar^2}{8m} \sum_i^N \nabla_{R_i}^2 \quad (32a)$$

and

$$H(r) = -\frac{\hbar^2}{2m} \sum_i^N \nabla_{r_i}^2 + \sum_{i<j} [V_{ij}^R(r) + V_{ij}^A(r)]. \quad (32b)$$

We note that  $H(R) + H(r)$  defines  $H(N)$  (Equation 1). Since we have  $V_{ij}^R(r) (\approx V_{HC}(r) \equiv A\tilde{\alpha}(r_{ij}))$  and for the SMW pair configuration we have shown<sup>55</sup> that  $\langle A\tilde{\alpha}(r_{ij}) \rangle = 0$ , we are, naturally, left with only  $V_{ij}^A(r)$  as the significant part of  $H(r)$ . One may easily find that the basic role of  $V_{ij}^A(r)$  is to decide the binding energy per particle ( $-V_0$ ) in N- as well as S-phases and to keep particles confined to volume  $V$ . However, as explained below,  $V_{ij}^A(r)$  in S-phase, where the WPs of neighboring particles tend to overlap, renders an *additional fall in the energy* of the system.

Since  $V_{ij}^A(r)$  perturbs only the S-state where each particle is represented only by its  $q$  motion of energy  $\epsilon_0$ , we can exclude thermal distribution of particles over the possible states of  $K$  motion and apply the perturbation theory of degenerate state. We construct a  $N \cdot N$  matrix of the expectation value of  $H(r)$  using  $\psi(r) = \sin(q_0 \cdot r)\exp(-i\epsilon_0 t/\hbar)$  as a basis that represents a WP in S-state. We have  $H(r)_{mn} = \epsilon_0$  for  $m = n$ , and  $H(r)_{mn} = V^A(r)_{mn}$  for  $m \neq n$  with  $V^A(r)_{mn}$  having non-zero value only if  $m$  and  $n$  refer to two neighboring WPs. Note that each WP has 6–12 (depending on the symmetry of their assumed spatial arrangement) nearest neighbor WPs. The diagonalisation of this matrix renders  $N/2$  energy levels with energy  $> N\epsilon_0(T_\lambda)$  (*anti-bonding states*) and  $N/2$  energy levels with energy  $< N\epsilon_0(T_\lambda)$  (*bonding states*). The system obviously falls in the lowest possible energy state, *i.e.*, a *bonding state*. As the perturbative effect of  $V_{ij}^A(r)$  lowers  $\epsilon_0$  of each particle identically, all particles in the system fall in bonding state simultaneously and acquire a kind of collective binding. Using the coherence property of the system evident from  $\Delta\phi = 2n\pi$  we find that the effective binding per particle becomes much larger than  $K_B T^{66}$ , even when the real binding per particle is very small. This ensures the stability of the bonding state in spite of its source being an energetically weak effect. Further since the WPs in the system at  $T \leq T_\lambda$ , always tend to acquire increased size beyond its value at  $T_\lambda$  (*i.e.*,  $\lambda/2 = d_\lambda$ ), they can not have a size  $< d_\lambda$  to come out of the bonding. This again ensures the stability of bonding state. The effect can also be understood in terms of Feynman's useful theorem<sup>63, p. 273</sup> used to explain stability of a superconducting state.

**5.4.3: Energy gap.** In view of the above discussion, we note that the  $q$  motion energy of the system falls from  $N\epsilon_0(T_\lambda)$  to a new value  $N\epsilon_0(T) = N\epsilon_0(T_\lambda) - N|v_N(T)| = N\epsilon_0(T_\lambda) - E_g(T)$  when the system is cooled below  $T_\lambda$ . This gives a simple method of finding  $|v_N(T)|$  (a measure of the net perturbative effect of attraction) through

$$E_g(T) = N|v_N(T)| = N[\epsilon_0(T_\lambda) - \epsilon_0(T)] \approx Nh^2(d_T - d_\lambda)/4md_\lambda^3. \quad (33)$$

Following this analysis,  $E_g(T)$  can be identified as: (i) *the collective binding* among all particles rendering the system to become a kind of single molecule<sup>67</sup> and (ii) an energy gap between the S and N phases in a sense that S becomes N phase if  $E_g(T)$  energy is supplied from outside. Although, our  $(\mathbf{q}, -\mathbf{q})$  pair appears to be similar to a Cooper pair<sup>68</sup>, it differs from the latter for the facts that

(i) the binding of two bosons is a consequence of their inherent attraction combined with QCP (*cf.* Section 3.0) and (ii) each particle of the pair represents the  $(\mathbf{q}, -\mathbf{q})$  pair. The system retains its fluidity because  $|v_N(T)| \ll \varepsilon_0$ , which also implies that the binding is a *weak effect*.

**5.4.4: Self energy of particles.** When the system moves to lower energy state the WPs have increased size depending on  $|v_N(T)|$ . This forces the system to expand with decreasing  $T$  which is corroborated by experimentally observed negative volume expansion coefficient in case of LHE-4. The expansion forced against inter-atomic attraction needs energy which should be managed from within the system. In this context we note that even in its S-phase the system has a small number  $N^*(T)$  of thermally excited particles (devoid of quantum correlation for their excitation wavelength  $\Lambda < \sigma$ ). With fall in  $T$  below  $T_\lambda$ , the number of these particles decreases from  $N^*(T_\lambda)$  to  $N^*(T)$  and in this process  $q$  motion energy decreases additionally by

$$\Delta\varepsilon(T) = k_B T_0 \ln 2 [N^*(T_\lambda) - N^*(T)], \quad (34a)$$

with

$$N^*(T) = \frac{V}{4\pi^2} \left[ \frac{2m}{\hbar^2} \right]^{3/2} \int_{\varepsilon_c}^{\infty} \left[ \exp\left(\frac{\varepsilon - \varepsilon_0}{k_B T}\right) - 1 \right]^{-1} \sqrt{\varepsilon} d\varepsilon, \quad (34b)$$

where  $\varepsilon_c = \hbar^2 Q_c^2 / 2m$  (with  $Q_c \approx 2\pi/\sigma$ ) represents such an energy that an atom of  $\varepsilon > \varepsilon_c$  has no quantum correlation with its neighbors. Note that Equation 34b gives an approximate  $N^*(T)$ , since in writing this relation we used a free particle dispersion,  $\varepsilon = \hbar^2 Q^2 / 2m$ , which is valid only to a good approximation. The fact that  $\Delta\varepsilon(T)$  (Equation 34a) and  $E_g(T)$  (Equation 33) closely satisfy  $\Delta\varepsilon(T) = E_g(T)$  has been observed for *He-II* for all  $T < T_\lambda$  (*cf.* Section 7.0). As such the  $\Delta\varepsilon(T)$  energy released from quantum correlations becomes available for the expansion of the system wherein particles are pushed to higher potential by an amount,  $\Delta V_s(T) = \Delta\varepsilon(T)$ . Naturally,  $\Delta V_s(T)$  represents a kind of strain in the system and it can be termed as self energy of the particles.

Since  $\Delta V_s(T)$  depends on the size of WPs and hence on  $q$  values, it serves as a source of OMONs (*collective oscillations of q*). It can also be recognized as the energy of omon field. The fact that  $\Delta V_s(T)$  increases with decreasing  $T$  implies that omon field intensity increases when phonon field intensity decreases. Evidently,  $\Delta V_s(T)$  could either be considered as the energy of phonons absorbed by the system or the omon be identified as an anti-phonon quantum quasi-particle. Since  $\Delta V_s(T)$  attains its maximum value at  $T = 0$ , it serves as a source of energy for collective motions even at  $T = 0$ .

**5.4.5: Bound pair exists in S-phase only.** Since the average WP size in the system at  $T > T_\lambda$  is smaller than  $d$ , the WPs in their spatial arrangement are not forced to have any overlap, which implies  $v_N(T) = 0$ . However, the WPs in the system below  $T_\lambda$  tend to have a size larger than  $d$  by forcing the system to expand. This forces each WP to overlap with its neighboring WPs and we have

$v_N(T) \neq 0$ . Evidently, bound ( $\mathbf{q}$ ,  $-\mathbf{q}$ ) pairs exist only in S-phase and not in N-phase.

It should be noted that our *bound pair* represents two particles bound in  $\phi$ -space (*i.e.*, locked at  $\Delta\phi = 2n\pi$ ). While the impact this binding can be observed in  $r$ -space due to  $\phi = kr$ , however, this does not imply that two particles in the system form  $He_2$  type diatomic molecule; the entire system assumes a state where particles interact with their neighbors identically. We use the term bound pair because the SMW quantum state of two particles is a result of the superposition of two plane waves of  $\mathbf{q}$ , and  $-\mathbf{q}$  momenta and the energy of such a state in the S-phase is lower than that in N-phase.

### 5.5: Energy Gap and its Consequences

In what follows from Appendix A,  $F(K)$  accounts for the routine thermodynamic properties, while  $E_g(T) = N\epsilon_0 - F(q)$  (*cf.* Equation A-2) forms an important component of  $F$  needed to explain superfluidity and related properties. In the following we, therefore, study all these aspects using  $E_g(T)$ .

*5.5.1: Superfluidity and related properties.* If two heads  $X$  and  $Y$  in the system have small  $T$  and  $P$  (pressure) differences, the equation of state is  $E_g(X) = E_g(Y) + S\Delta T - V\Delta P$ . Using  $E_g(X) = E_g(Y)$  for equilibrium, we get

$$S\Delta T = V\Delta P \quad (35)$$

This reveals that (i) the system should exhibit thermo-mechanical and mechano-caloric effects, and (ii) the measurement of  $\eta$  by capillary flow method performed under the condition  $\Delta T = 0$  and of thermal conductivity ( $\Theta$ ) determined under  $\Delta P = 0$  should reveal  $\eta = 0$  and  $\Theta \approx \infty$ , respectively. As such, the S-phase is expected to be a superfluid of infinitely high  $\Theta$ .

Our theory also provides a good understanding of the above inferred behavior from a phenomenological point of view. In this context we note the following: (i) A close packed arrangement of WPs in a fluid-like system can have no vacant site particularly because two neighboring particles experience zero point repulsion. The system is, naturally, expected to have large  $\Theta$ . (ii) The fact that the system can not have thermal convection currents for its large  $\Theta$  and close packing of particles, explains why *He-II* does not boil like *He-I*. (iii) Since particles in S-phase can move only in the order of their locations (*cf.* Section 5.1), they cease to have relative motion, particularly during their linear motion and we have vanishingly small  $\eta$ . In the rotating fluid, however, particles moving on the neighboring concentric circular paths have relative velocity producing quantized vortices as a source of natural viscous behavior. This explains both *viscosity and rotation paradoxes*<sup>5</sup>. As such, the loss of viscosity in linear motion is not due to any loss of viscous forces among the particles; rather it is the property of the S-phase configuration (*i.e.*, close packed arrangement of WPs) originating mainly from the wave nature of particles.

*5.5.2: Critical velocities and stability of S-phase.* Using the same argument,

which gave us Equation 30, we find that the state function  $\phi_n(q_0)$  of the S-phase changes to  $\phi_n^*(q_0)$  when the system flows with velocity  $v_f = \hbar \Delta \mathbf{q} / m$ . We have

$$\phi_n^*(q_0) = \phi_n(q_0) \exp \left[ i \mathbf{K} \cdot \sum_i^N \mathbf{R}_i \right] \exp[-i[N(\varepsilon_0 + \varepsilon(K)) - E_g(T)]t/\hbar], \quad (36)$$

with  $2\Delta \mathbf{q} = \mathbf{K}$ . This reveals that the S-state function remains stable against such flow unless the flow energy  $Nmv_f^2/2 = N\mathcal{E}(K)$  overtakes the collective binding energy  $E_g(T)$  of the system. This explains the origin for critical velocities. Equating  $E_g(T)$  and flow energy,  $Nmv_f^2/2$  for  $v_f = v_c$ , we get upper bound of *critical velocity*,  $v_c$ , for which the S-phase becomes N-phase. We have

$$v_c(T) = \sqrt{[2E_g(T)/Nm]}. \quad (37)$$

$Av_c < v_c(T)$ , at which the superfluid may show signs of viscous behavior, can be expected due to creation of quantized vortices. However, this cause would not destroy superfluidity unless energy of all vortices produced in the system exceeds  $E_g(T)$ .

**5.5.3: Coherence length.** The main factors responsible for the coherence of the S-phase are the locking of particles at  $\Delta\phi = 2n\pi$  (cf. Equation 20) and their collective binding  $E_g(T)$ . Naturally, the *coherence length*, (not to be confused with healing length<sup>5</sup>), can be obtained from

$$\xi(T) = 1/mv_c(T) = h\sqrt{[N/2mE_g(T)]}. \quad (38)$$

**5.5.4: Superfluid density.** Correlating the *superfluid density*,  $\rho_s$ , as the order parameter of the transition, with  $E_g(T)$  we find a new relation

$$\rho_s(T) = [E_g(T)/E_g(0)]\rho(T) \quad (39)$$

to determine  $\rho_s(T)$  and *normal density*,  $\rho_n(T) = \rho(T) - \rho_s(T)$ . Evidently,  $v_c(T)$ ,  $\xi(T)$ , and  $\rho_s(T)$  can be obtained if we know  $E_g(T)$  (Equation 33), which requires  $\rho(T)$  values. Finally it may be mentioned that  $E_g(T)$  as well as  $\rho_s(T)$  would vanish at the boundaries of the system, since S-state function vanishes there.

**5.5.5: Superfluid velocity.** Concentrating only on the time independent part, Equation 36 can be arranged as

$$\phi_n^*(q_0) = \phi_n(q_0) \exp[iS(R)], \quad (40a)$$

with

$$S(R) = \mathbf{K} \cdot \sum_i^N \mathbf{R}_i \quad (40b)$$

and

$$v_s = \frac{\hbar}{2m} \nabla_{R_j} S(R) = \frac{\hbar \Delta \mathbf{q}}{m}, \quad (40c)$$

which follows the fact that  $\nabla_{R_j} S(R)$  renders the momentum of the pair (not of a single particle). Evidently Equation 40c shows the interrelationship of the superfluid velocity  $v_s$  and the phase  $S(R)$  of our S-state wave function as expected (cf. Section 2.3 of reference 8).

### 5.6: Quantized Vortices

Using the symmetry property of a state of bosons, Feynman<sup>53,63</sup> showed that the circulation,  $\kappa$ , of the velocity field should be quantized, i.e.,  $\kappa = nh/m$  with  $n = 1, 2, 3, \dots$ . However, Wilks<sup>1</sup> has rightly pointed out that this account does not explain the fact that *He-I*, to which Feynman's argument applies equally well, does not exhibit quantized vortices. Using Equation 40c for the superfluid velocity  $v_s$ , we find that

$$\kappa = \sum_i \mathbf{v}_s(i) \cdot \Delta \mathbf{r}_i = \frac{\hbar}{m} \sum_i \Delta \mathbf{q}_i \cdot \Delta \mathbf{r}_i = \frac{nh}{m}, \quad (41)$$

where we used the condition that  $\sum_i \Delta \mathbf{q}_i \cdot \Delta \mathbf{r}_i = 2n\pi$ , which presumes that the relative configuration of particles during their reshuffle on a closed path maintain phase correlation. However, we find the particles of S-phase only have their  $\phi$ -positions locked at  $\Delta\phi = 2n\pi$  and for this reason we observe quantized vortices in this phase. On the other hand particles in N-phase having random distribution ( $\Delta\phi \geq 2n\pi$ ) in  $\phi$ -space do not sustain phase correlation and we do not observe quantized vortices.

### 5.7: Single Particle Density Matrix and ODLRO

Using Equations 15 and 10, we can have the single particle density matrix,

$$\begin{aligned} \rho(\mathbf{R}^* - \mathbf{R}) &= \left[ \frac{N_{K=0}(T^*)}{V} + \frac{N}{\lambda_T^3} \exp \left[ -2\pi \frac{|\mathbf{R}^* - \mathbf{R}|^2}{\lambda_T^2} \right] \right] \\ &\times \left( \frac{2}{V} \sin^2 \left[ \frac{\pi(r'' - r')}{d} \right] \right), \end{aligned} \quad (42a)$$

with

$$n_{K=0}(T^*) = \frac{N_{K=0}(T^*)}{N} = \left[ 1.0 - \left( \frac{T^*}{T_\lambda^*} \right)^{3/2} \right] = \left[ 1.0 - \left( \frac{T - T_0}{T_\lambda - T_0} \right)^{3/2} \right], \quad (42b)$$

where we used (i)  $\mathbf{q}_0 \cdot (\mathbf{r}'' - \mathbf{r}')^* - 2n\pi + \mathbf{q}_0 \cdot (\mathbf{r}'' - \mathbf{r}')$ , (ii) a renormalized  $T$  scale by defining  $T^* = T - T_0$  since  $T^* = 0$  represents a state where  $K$  motions of the system are expected to freeze at zero level and (iii) the standard relation,



$$n_{K=0}(T) = \frac{N_{K=0}(T)}{N} = \left[ 1 - \left( \frac{T}{T_\lambda} \right)^{3/2} \right] \quad (42c)$$

available from the theory of BEC of non-interacting bosons<sup>5</sup>. Use of Equation 42c can be justified since plane wave  $K$  motions in the system represent a kind of non-interacting bosons. While the term in big (...) of Equation 42a represents the variation of density over a single AR,  $N_{K=0}(T^*)$  stands for the number of particles condensed to the state of  $K = 0$  and  $q = \pi/d$ ;  $\lambda_T^i = h / [2\pi(4m)k_B T]^{1/2}$  represents thermal wavelength attributed to  $K$  motions. We note that under the limit  $|\mathbf{R}^* - \mathbf{R}|$  tends to  $\infty$ , the “one particle density matrix” ( $\rho(\mathbf{R}^* - \mathbf{R})$ ) has nonzero value ( $N_{K=0}(T)/V$ ) for  $T < T_\lambda$  and zero for  $T \geq T_\lambda$  since  $N_{K=0}(T^*)$  is  $\neq 0$  for  $T < T_\lambda$  and 0 for  $T \geq T_\lambda$ . Evidently, our theory satisfies the criterion of Penrose and Onsager<sup>35</sup> for the occurrence of BEC in the G-state of the system defined by  $K = 0$  and  $q = \pi/d$  and agrees with the idea of ODLRO, spontaneous symmetry breaking and phase coherence advanced, respectively, by Yang<sup>37</sup>, Goldstone<sup>69</sup> and Anderson<sup>70</sup>.

### 5.8: Logarithmic Singularity of Specific Heat

The specific heat  $C_p(T)$  of the system is expected to show usual cusp at  $T_\lambda$  if BEC of SMW pairs is considered as the only mechanism of the transition. But our system at its  $\lambda$ -point also has an onset of ordering of particles in phase space rendering widely different changes in  $\phi$ -position of different particles. To determine the corresponding change in energy  $\Delta\mathcal{E}$  we, however, assume for simplicity that of the  $N^*(T_\lambda)$  uncorrelated particles in their excited states,  $N_\lambda$  make significant contribution to  $\Delta\mathcal{E}$  and they move from their  $\phi = (2n\pi \pm \delta\phi_\lambda)$  at  $T_\lambda^+$  (just above  $T_\lambda$ ) to  $\phi = (2n+1)\pi$  at  $T_\lambda^-$  (just below  $T_\lambda$ ). This gives

$$\Delta\mathcal{E} = -N_\lambda k_B T_0 \left[ \ln 2 \sin^2 \left( \frac{2n\pi \circ \delta\phi_\lambda}{2} \right) - \ln 2 \right]. \quad (43)$$

Following the theories<sup>71</sup> of critical phenomenon we may define

$$\delta\phi_\lambda = \delta\phi_\lambda(0)|\zeta|^\nu [1 + a_2|\zeta|^2 + a_3|\zeta|^3] \quad (44)$$

with  $\zeta = (T - T_\lambda)/T_\lambda$ . To a good approximation we have

$$\Delta\mathcal{E} = -N \left( \frac{T - T_\lambda}{T_\lambda} \right) k_B T_0 \ln \left( \frac{\delta\phi_\lambda(0)|\zeta|^\nu}{2} \right)^2 \quad (45)$$

by using  $\delta\phi_\lambda = \delta\phi_\lambda(0)|\zeta|^\nu$  and  $N_\lambda = N(T - T_\lambda)/T_\lambda$ ; the latter expression is so chosen to ensure that  $\Delta\mathcal{E}$  does not diverge at  $T_\lambda$  and it decreases with decreasing  $T$  through  $T_\lambda$ . Equation 43 gives

$$C_p(T \approx T_\lambda) \approx -\frac{N}{T_\lambda} k_B T_0 [2\nu \ln |\zeta| + \ln(\delta\phi_\lambda(0)^2) - \ln 4 + 2\nu]. \quad (46)$$

### 5.9: S-State and its Similarity with Lasers

We note that the system below  $T_\lambda$  defines a 3-D network of SMWs extending from its one end to another without any discontinuity. In lasers too these are the standing waves of electromagnetic field that modulate the probability of finding a photon at a chosen phase point. The basic difference between the two lies in the number of bosons in a single AR. In the case of lasers this could be any number since photons are non-interacting particles but for a SIB like  ${}^4\text{He}$  or  ${}^{87}\text{Rb}$  one AR can have only one atom.

### 6.0: Thermodynamic Behavior at $T > T_\lambda$

Following Equation 4, the energy of a particle in SMW configuration is given by

$$E = \varepsilon(K) + \varepsilon(k) = \frac{\hbar^2 K^2}{8m} + \frac{\hbar^2 k^2}{8m}. \quad (47a)$$

The possible values of  $E$  range between its G-state energy  $\varepsilon_0 = \hbar^2/8md^2 = E$  (for  $K = 0$  and  $k = 2\pi/d$ ) and 1 and this range is available if we write

$$E = \frac{\hbar^2 K^2}{8m} + \varepsilon_0, \quad (47b)$$

where  $K$  varies between 0 and 1. Using Equation 47b in the starting expressions of the standard theory of BEC<sup>72</sup>, we have

$$\frac{PV}{k_B T} = - \sum_{\varepsilon(K)} \ln[1 - z \exp(-\beta[\varepsilon(K) + \varepsilon_0])] \quad (48)$$

and

$$N = \sum_{\varepsilon(K)} \frac{1}{z^{-1} \exp(\beta[\varepsilon(K) + \varepsilon_0]) - 1}, \quad (49)$$

with  $\beta = 1/k_B T$  and fugacity

$$z = \exp(\beta\alpha) \quad (\alpha = \text{chemical potential}) \quad (50)$$

Following the steps of the standard theory of BEC<sup>72</sup> and by redefining the fugacity by

$$z' = z \cdot \exp(-\beta\varepsilon_0) = \exp[\beta(\alpha - \varepsilon_0)] \quad (51)$$

we easily have

$$\frac{P}{k_B T} = - \frac{2\pi(8mk_B T)^{3/2}}{h^3} \int_0^\infty x^{1/2} \ln(1 - z' e^{-x}) dx = \frac{1}{\lambda^3} g_{5/2}(z'), \quad (52)$$

and

$$\frac{N - N_0}{V} = \frac{2\pi(8mk_B T)^{3/2}}{h^3} \int_0^\infty \frac{x^{1/2} dx}{z'^{-1} e^x - 1} = \frac{1}{\lambda^3} g_{3/2}(z'), \quad (53)$$

where  $x = \beta \varepsilon(K)$ ,  $\lambda = h/(2\pi(4m)k_B T)^{1/2}$  and  $g_n(z')$  has usual expression. This reduces our problem to that of non-interacting bosons but with a difference. Note that  $z' = 1$  expected for  $T \leq T_\lambda$  implies  $\alpha = \varepsilon_0$  (cf. Equation 51), and  $z' < 1$  for  $T > T_\lambda$ , demands  $\alpha < \varepsilon_0$  but for a system of non-interacting bosons we have  $\alpha = 0$  for  $T \leq T_\lambda$ , and  $\alpha < 0$  for  $T > T_\lambda$ .

The expressions for different thermodynamic properties of the N-phase of the system (at  $T$  near  $T_\lambda$ ) should, obviously, not be different from those derived for non-interacting bosons<sup>72</sup> except that  $m$  is replaced by  $4m$ . This justifies our relations (cf. Equations 19a and 19b) for  $T_\lambda$ . The accuracy of such expressions depends on the validity of Equation 47b, which assumes that average  $\varepsilon(k) = \varepsilon_0$  and implies that the system is a close packed arrangement of WPs of an average size  $d$ , which in turn reveals that the velocity ( $v_s$ ) of long wave length sound modes should satisfy  $v_s = v_g = h(\pi)^{1/2}/2md$  (Equation 28). We note that the experimental  $v_s$  for *He-I* really satisfies  $v_s = v_g$  closely at least for the temperature range  $T_\lambda$  to  $T = 3.2 \text{ K}$ . This means that Equation 47b holds at  $T > T_\lambda$  to a good approximation. As such the thermodynamic properties of the N-phase basically arise from  $K$  motion. Solving for the internal energy of the system  $U = -(\partial/\partial\beta)(PV/k_B T)_{\alpha, V}$  by using Equations 52 and 53, we have  $U = (3/2) \cdot k_B T \cdot (V/\lambda^3) g_{5/2}(z') + N\varepsilon_0 = U' + N\varepsilon_0$ , where  $U' = -(\partial/\partial\beta)(PV/k_B T)_{\alpha, V}$  represents the contribution of  $K$  motions and  $N\varepsilon_0$  (=zero point energy) that of  $k$  motions. This analysis not only justifies Equations 48 and 49 but also establishes that the process of BEC of particles in the state of  $K = 0$  and  $q = \pi/d$  should explain the thermodynamic behavior of the N-phase of the system near  $T_\lambda$ . This is corroborated by the fact that specific heat of *He-I* ( $\approx 1.7 k_B$ ) over the  $T$  range from 2.3 to  $\approx 3.2 \text{ K}$  is very close to its expected value (i.e., slightly higher than  $1.5 k_B$ <sup>72</sup>). For better accuracy of the results one may also include the  $T$  dependence of the constant negative interaction energy term assumed to be  $T$  independent in this analysis.

## 7.0: Properties of He-II

### 7.1: Thermodynamic Properties

In view of the observation<sup>4</sup> that  $E(Q)_{\text{Feyn}}$  (Equation 29) obtained by Feynman<sup>53,63</sup> renders  $E(Q)$  values nearly two times the experimental  $E(Q) \equiv E(Q)_{\text{expt}}$  for *He-II*, our theoretical relation (Equation 31) should match closely with  $E(Q)_{\text{expt}}$ . Naturally, this ensures that our theory can explain the thermodynamic properties of *He-II* accurately. Further, since our theory also provides an alternative set of relations (Equations 23–27) to obtain  $E(Q)$  of SIB, we also used these relations to obtain  $E(Q)$  of *He-II* by using three adjustable parameters  $C(Q)$ ,  $d(Q)$  and  $m_f$ . We used Equation 23 to obtain  $E(Q \leq Q_{\text{max}} = \pi/\sigma)$  by

employing

$$d(Q) = d - d' \sin \pi Q/2Q_{\max} = 3.5787 - 0.7484 \sin \pi Q/2.22 \quad (54)$$

and

$$C(Q) = C + C' \sin \pi Q/2Q_{\max} = 3.1681 + 2.3619 \sin \pi Q/2.22 \quad (55)$$

to find necessary  $d(Q)$  and  $C(Q)$  at different  $Q \leq Q_{\max}$ . These relations have been constructed intuitively to ensure smooth variation of  $d(Q)$  and  $C(Q)$  from their values at  $Q = 0$  to  $Q = Q_{\max}$  without any discontinuity even in their first  $Q$  derivative at  $Q_{\max}$  beyond which they become  $Q$  independent. Values of  $d$ ,  $d'$ ,  $C$  and  $C'$  were fixed empirically by using certain experimental data for *He-II*. We used (i) density of *He-II* ( $0.1452 \text{ gm/cc}$ )<sup>1,5</sup> to fix the desired parameters of Chain-A (Figure 1), *i.e.*,  $d(Q = 0)$  and  $C(Q = 0) = 3.1681 \text{ dyne/cm}$  by using Equation 22, (ii) experimental  $E(Q)$  of *He-II* to find similar parameters of Chain-B, *i.e.*,  $d(Q_{\max}) = \sigma = 2.8293 \text{ \AA}$  from the maxon position  $Q_{\max} = \pi/\sigma = 1.11 \text{ \AA}^{-1}$  and  $C(Q_{\max}) = 5.53 \text{ dyne/cm}$  by using  $E(Q_{\max}) = 13.92 \text{ K}^4$ , and used these results to fix  $d' = d - \sigma = 0.7484 \text{ \AA}$  and  $C' = C(Q_{\max}) - C(Q = 0) = 2.3619 \text{ dyne/cm}$ .

The fact that our  $E(Q)_{\text{theor}}$  at low  $Q < \pi/\sigma$  has anomalous character is evident from Figure 2 where we depict our calculated  $v_p(Q)$  and  $v_g(Q)$  which rise to a maximum of  $\approx 5\text{--}6 \text{ m/sec}$  above  $v_p(Q = 0)$  and  $v_g(Q = 0)$ . The fact that this agrees closely with their experimental rise of  $\approx 9 \text{ m/sec}$ <sup>9</sup> concludes the accuracy of our  $E(Q \leq Q_{\max})$  values obtained from Equation 23.

We used Equation 24 to find  $E(Q_{\max} \leq Q \leq 2\pi/d)$  by using  $d(Q_{\max}) = \sigma$ , and  $C(Q_{\max}) = 5.53 \text{ dyne/cm}$  and Equation 25 to find  $E(Q \geq 2\pi/\sigma)$  where a good match between theory and experiment could be obtained by varying  $m_F$  between  $3.3m_{He}$  at  $Q = 2\pi/\sigma$  and  $m_F \approx m_{He}$  at  $Q \geq 3.0 \text{ \AA}^{-1}$ . In view of the fact that the nature of the excitation at  $Q = 2.22 \text{ \AA}^{-1}$  starts changing from the correlated motion of SMW pair to a single particle motion, the  $m_F \approx 3.3m_{He}$  at this  $Q$  closely agrees with the inference of our theory that a particle in SMW pair configuration moves as free particle of mass  $4m$ . When this particle moves out of the pair correlated motion, it assumes a state of freely moving particle with increasing  $Q$  and we really find  $m_F \approx m$  for  $Q \geq 3.0 \text{ \AA}^{-1}$ .

Interestingly, position and energy of roton minimum  $Q_{\min} = 1.99 \text{ \AA}^{-1}$  and  $E_{\text{rot}}(Q_{\min}) \approx 8.31 \text{ K}$ , as obtained from Equations 26 and 27c, are found to be nearly equal to their experimental values, *viz.*  $Q_{\min} = 1.92 \text{ \AA}^{-1}$  and  $E_{\text{rot}}(Q_{\min}) = 8.65 \text{ K}$ . Evidently, the  $E(Q)_{\text{theory}}$  has an overall agreement with  $E(Q)_{\text{expt}}$  for *He-II*. Our detailed study of  $E(Q)$ <sup>61,62</sup> of *He-II* (*to be published separately*) concludes that its plateau ( $E(Q) \approx 18\text{--}20 \text{ K}$  for  $Q > 3.0 \text{ \AA}^{-1}$ ) arises due to a resonance interaction between multiphonon branch ( $E(Q) \approx 20 \text{ K}^4$ ) and  $E_{\text{sp}}(Q)$  (Equation 25). If this resonance interaction were absent the  $E(Q)$  is expected to follow  $E_{\text{sp}}(Q)$  (Equation 25) as shown by the dotted part in Figure 1. As such,

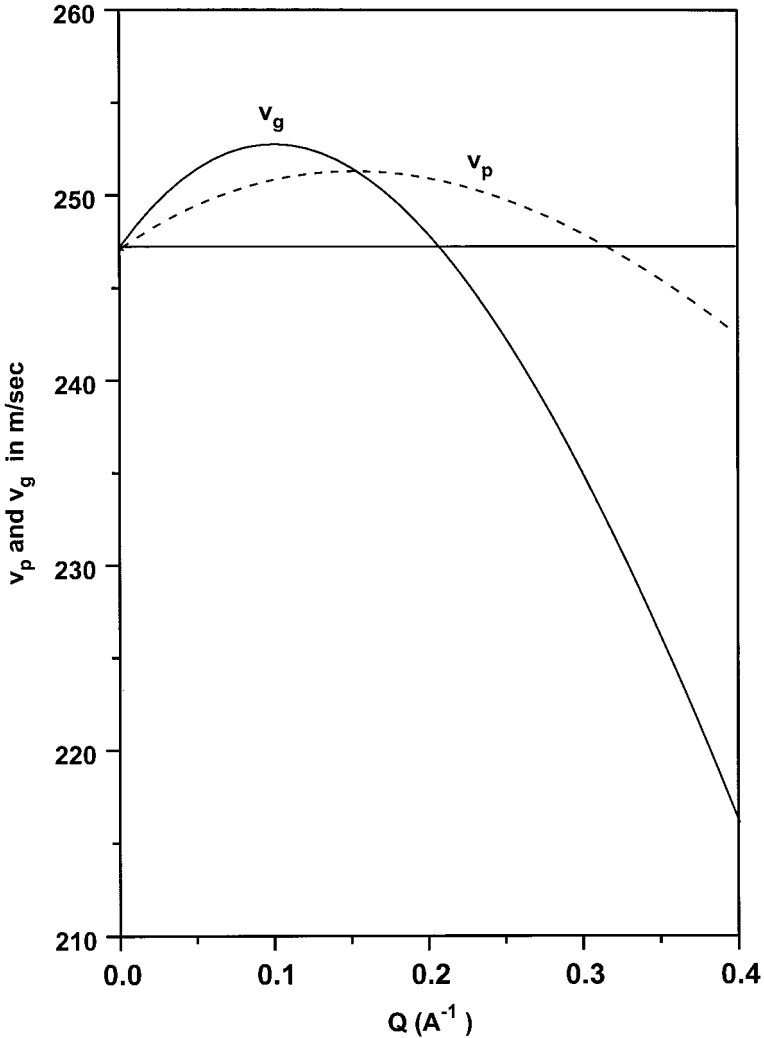


Fig. 2. Anomalous nature of  $v_p(Q)$  and  $v_g(Q)$  of *He-II* obtained by using  $d(Q \leq Q_{\max}) = 3.5787 - 0.7484 \sin(\pi Q/2.22)$  and  $C(Q \leq Q_{\max}) = 3.3181 + 2.3619 \sin(\pi Q/2.22)$  in Equation 23. For details see Sections 5.2 and 7.0.

for the first time, our model presents a clear picture of microscopic details of collective motions of the system.

The most important aspect of our theory is its capacity to explain the experimentally observed logarithmic singularity in  $C_p(T)$  at  $T_\lambda$  which remained unexplained as yet. The problem of explaining this singularity was considered to be a challenging task by Feynman as indicated in his book<sup>63, p.34</sup> but the present theory reveals its details in good agreement with experimental results for liq-

uid  ${}^4\text{He}$ . Using the parameters of liquid  ${}^4\text{He}$  and  $\nu = 0.55$  and  $\delta\phi_\lambda(0) = \pi$  in Equation 46, we find

$$C_p(\text{J/mole} \cdot \text{K}) \approx -5.71 \ln|\zeta| - 10.35 = -A \ln|\zeta| + B, \quad (56)$$

while the experimental results reveal  $A = 5.355$  and  $B = -7.77$  for  $T > T_\lambda$  and  $A = 5.1$  and  $B = 15.52$  for  $T < T_\lambda^{73}$ . The fact that our  $A$  value agrees closely with experiments speaks of the accuracy of our theoretical result. With respect to our choice of  $\nu = 0.55$  and  $\delta\phi_\lambda(0) = \pi$ , we note that (i)  $\delta\phi_\lambda$  originates basically from change in momentum  $\Delta k \approx \xi^{-1}$  and  $\xi$  varies around  $T_\lambda$  as  $|T - T_\lambda|$ ; note that critical exponent  $\nu$  lies in the range 0.55 to 0.7<sup>71</sup>. However, we have no definite reason for our choice of  $\delta\phi_\lambda(0) = \pi$ , except that  $\pi$  is the largest possible value by which phase position of a particle can change.

### 7.2: Hydrodynamical Properties

To prove that our theory explains the hydrodynamic properties of *He-II*, we note that it vindicates: (i) two fluid theory of Landau (Section 5.3), (ii) presence of quantized vortices (Section 5.6) and (iii) vanishing of  $\rho_s(T)$  at the boundaries (Section 5.5.4) of the system. We also find that our theoretical  $\rho_s(T)$  matches closely with its experimental values for *He-II*; this is evident from Figure 3 where we depict: (i) our theoretical  $\rho_s(T)$  (*cf.* Curve-A) obtained from  $\Delta\mathcal{E}(T) = E_g(T)$  by using Equations 34 and 39 and (ii) experimental  $\rho_s(T)$  (*cf.* Curve-E1) derived from  $T$  dependence of the experimental  $\rho$  of *He-II* available from reference 1 by using Equations 33 and 39 and similar results (*cf.* Curve-E2) obtained from second sound experiments reported in reference 5.

In Figure 3 we also depict  $n_{K=0}(T^*)$  (Equation 42b) and  $n_{K=0}(T)$  (Equation 42c) (*cf.* Curve-B\* and Curve-B). It is interesting to note that experimental  $\rho_s(T)$  matches closely with  $n_{K=0}(T^*)$  (Curve-B\*) not with  $n_{K=0}(T)$  (Curve-B); this corroborates our inference that  $K = 0$  condensation takes place in the G-state of energy  $\mathcal{E}_0$  equivalent of  $T_0$ .

### 7.3: Other Properties

Figure 3 also shows the  $T$  dependence of  $0.2 \cdot v_c(T)$  with  $v_c(0) = 8.46$  m/sec and  $10 \cdot E_g(T)$  with  $E_g(0) = .142$  J/Mole.  $E_g(0) \equiv \Delta\mathcal{E}(0)$  obtained from Equation 34a is set equal to  $E_g(0)$  obtained from Equation 33 by using  $\mathcal{E}_c = 10.35$  K in Equation 34b. Our estimates<sup>74</sup> of (i)  $T_\lambda \approx 2.26, 2.03,$  and  $1.96$  K, respectively, for *sc, bcc* and *fcc* assumed arrangement of WPs, (ii) upper bound  $v_c(T)$  changing smoothly from 0 (at  $T_\lambda$ ) to  $\approx 9$  m/sec (at  $T = 0$ ) (*cf.* Figure 3), (iii) the low  $Q$  values of  $v_p = v_g \approx 246$  (for *sc*), 238 (for *bcc*) and 220 (for *fcc*) m/sec, (iv)  $\xi(T)$  (Equation 38) varying smoothly from  $\approx 10^{-6}$  cm at  $T = 0$  to infinitely large value at  $T = T_\lambda$ , *etc.*, agree with experiments (see reference 5 for i–ii, reference 9 for iii, and reference 75 for iv).

While our value ( $\approx -21$  K) of potential energy per  ${}^4\text{He}^{61}$  atom does not differ

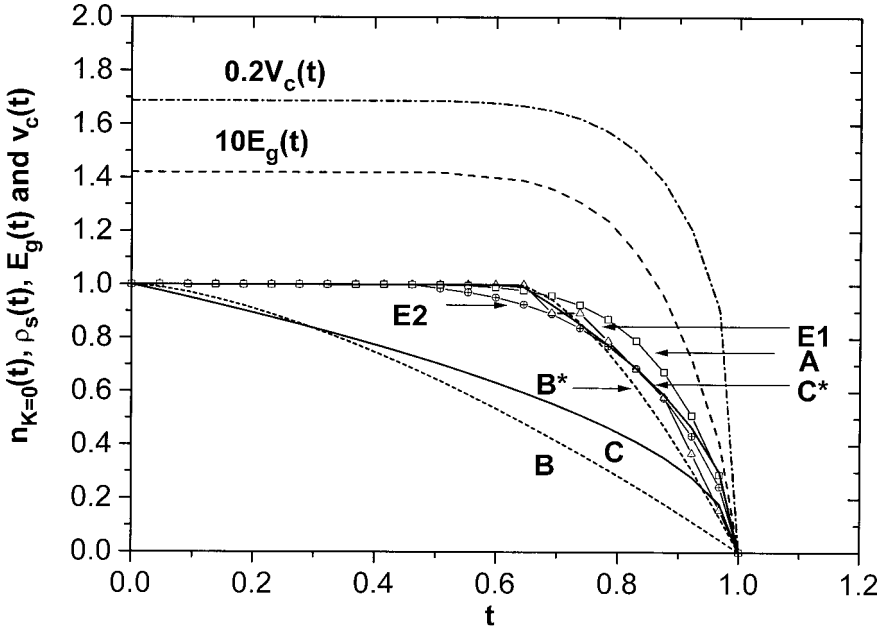


Fig. 3.  $t = T/T_\lambda$  dependence of (i)  $0.2.v_c(t)$  (m/sec), (ii)  $10.E_g(t)$  (J/mole), (iii)  $\rho_s(t)/\rho$ —Curve-A: Theoretical values using Equations 34 and 39 (with  $(\Delta\epsilon T) = E_g(T)$ ), and Curve-E1: Experimental values obtained from experimental density data<sup>1</sup> used in Equation 33, and Curve-E2: Experimental values obtained from second sound velocity, and (iv) Condensate Fraction or Order Parameter—Curve-B:  $n_{K=0}(T)$  (Equation 42c), Curve-B\*:  $n_{K=0}(T^*)$  (Equation 42b), Curve-C:  $n_{K=0}(T)$  (Equation A-7) and Curve-C\*:  $n_{K=0}(T^*)$  (Equation A-8). Curve-A also depicts  $n^*(T)$  (Equation A-9) which also defines an equivalent of  $n_{K=0}(T^*)$ .

from others<sup>18</sup>, our zero point kinetic energy ( $\approx 3.93$  K) is much lower than (14 K estimated by others<sup>18</sup>). Evidently, the configuration revealed by our theory is energetically favourable one. Using the values of  $\xi(T)$  and  $E_g(T)$  and following the standard method<sup>76</sup> we obtained the time of persistent currents for *He-II* as  $\approx 10^{34}$  sec, which is much larger than the life of our universe, *i.e.*,  $\approx 10^{18}$  sec. As such we find that all important conclusions of our theory that the S-state of the system should exhibit (i) negative volume expansion, (ii) infinitely high  $\Theta$ , (iii) coherence of particle motion, (iv)  $\eta = 0$  for its capillary flow, (v)  $\eta \neq 0$  in the state of its rotation, *etc.*, are well known properties of *He-II*.

An overall agreement between experimental and theoretical results for LHE-4 establishes the accuracy of our theory.

### 8.0: Comparison With CTs

CTs emphasize the classical size ( $\sigma$ ) of a particle, even for low energy particles of  $\lambda/2 > \sigma$  and use a boundary condition  $\psi_i(r \leq \sigma) = 0$  (or its equivalent). They *assume* that (i) BEC in a SIB is a collection of macroscopically large

number of atoms  $n_{p=0}(T)N$  in a single particle state of  $p = 0$ , (ii) the G-state of the system has  $n_{p=0}(0)N$  particles of  $p = 0$  at a single point of  $\phi = 0$  in  $\phi$ -space, and  $[1 - n_{p=0}(0)]N$  particles of  $p \neq 0$  at randomly distributed  $\phi$ -positions, and (iii)  $p = 0$  condensate, believed to be somehow related to  $\rho_s(T)$ , is the origin of superfluidity and related properties of the S-phase. The inter-particle repulsion,  $V_{ij}^R$ , is found to deplete  $n_{p=0}(T)$  to a large extent, *e.g.*, their calculations show that the maximum value of  $n_{p=0}(T = 0)$  in LHE-4 can be only  $\approx 0.13^{20}$ . CTs seek the origin of superfluidity and related properties in  $g(r)$  presumably because  $g(r)$  in S-phase differs slightly from that in N-phase and their presumed  $p = 0$  condensate is believed to be responsible for this difference. They further presume that to a good approximation  $T_\lambda$  is not different from  $T_{\text{BEC}}$  obtained for a SNIB<sup>72</sup>.

On the other hand our theory emphasizes the quantum size (or the WP size)  $\lambda/2$  of particles. It uses  $(\mathbf{q}, -\mathbf{q})$  pair condensation as its basic theme. By identifying the basic features of the relative motion of two HC particles, (*cf.* Equations 4 and 5 of reference 55), it obtains the correct wave function that represents two particles in a state of their wave mechanical superposition. It replaces the boundary condition,  $\psi_k(r \leq \sigma) = 0$ , losing its significance and meaning particularly for the particles of  $\lambda/2 > \sigma$ , by  $\lambda/2 \leq d$ , which ensures that the WPs of two particles do not overlap. The theory *concludes* that each particle in the system represents a  $(\mathbf{q}, -\mathbf{q})$  pair moving with CM momentum  $\mathbf{K}$  and  $\lambda$ -transition is the onset of an order-disorder of particles in  $\phi$ -space followed simultaneously by the BEC of  $(\mathbf{q}, -\mathbf{q})$  pairs in the G-state defined by  $q = \pi d$  and  $K = 0$ . As a basic feature of the S-phase, particles are found to have an orderly arrangement ( $\Delta\phi = 2n\pi$ ) in  $\phi$ -space. Superfluidity and related aspects are identified to be the obvious properties of the G-state configuration (Equation 20) and a *weak effect* which binds particles in  $\phi$ -space. The fact that the binding energy per <sup>4</sup>He atom in *He-II* is found to be about  $10^{-2}\epsilon_0$  speaks of the weakness of the effect. The inter-particle repulsion  $V_{ij}^R(r) \approx V_{\text{HC}}(r)$  forces all particles to condense into a single state of  $q = \pi d$  and  $K = 0$  leaving no chance for even one particle to have  $q < \pi d$  and the amount of  $K = 0$  condensate  $n_{K=0}(T^*)$  (Equation 42b) increases with decreasing  $T^*$  monotonically from  $n_{K=0}(T^*) = 0$  at  $T_\lambda$  to reach at exactly 1.0 at  $T^* = 0$ . We also find that another related identical quantity  $n^*(T)$  (Equation A-9, Appendix A), rises monotonically on cooling from  $n^*(T_\lambda) = 0$  to  $n^*(T = 0) = 1.0$  and this behavior of  $n^*(T)$  matches closely with experimentally observed  $\rho_s(T)$  in case of *He-II*. Concluding that particles in S-phase are dominated by phase correlations  $g(\phi)$  (a combination of  $q$ -correlations and  $r$ -correlations) as its additional aspect (not present in N-phase), our theory emphasizes  $g(\phi)$  as the origin of the unique properties of S-phase. For the first time our theory could find the origin of the experimentally observed logarithmic singularity of specific heat and related properties at  $T_\lambda$ . As such it finds a solution to the problem which at times was considered it to be a challenging problem by Feynman<sup>63, p. 34</sup>. Similarly, it also provides a more accurate account of the observation of quantized vortices in S-phase and their absence in N-phase.



It may, finally, be mentioned that pairing theories have also been developed earlier by using conventional approach by Valatin and Butler<sup>77</sup>, Girardeau and Arnowitt<sup>78</sup>, Luban<sup>79</sup>, Kobe<sup>80</sup>, and Brown and Coopersmith<sup>81</sup> assuming the existence of  $p = 0$  condensate, and by Congilio *et al.*<sup>82</sup>, and Evans and Imery<sup>83</sup> by incorporating an effective attraction that produces pair condensation in the system. However, these theories have been developed by usual conventional field theoretical methodology and they too fail to explain the properties of LHE-4.

### 9.0: Concluding Remarks

This paper presents a microscopic theory of a system of interacting bosons such as liquid  ${}^4\text{He}$ . It explains the properties of liquid  ${}^4\text{He}$  with unparalleled accuracy, simplicity and clarity. It is consistent with excluded volume condition<sup>51</sup> as well as the microscopic and macroscopic uncertainty<sup>5</sup>. It vindicates (i) the two fluid theory of Landau<sup>14</sup>, (ii) London's idea of macroscopic wave function of the S-state<sup>12</sup>, and (iii) the observation of Bogoliubov<sup>15</sup> that superfluidity is an interplay of inter-particle interactions, *etc.* All particles below  $\lambda$ -point have a kind of collective binding which serves as an energy gap between the S- and N-phases and makes the entire system behave like a single macromolecule as envisaged by Foot and Steane<sup>67</sup>.

When the interactions are switched off, a particle would not experience the existence of the other. We note that such a particle can only have its self superposition when it is reflected from the walls of the container and this state would not last long unless its  $\lambda \approx 2L$  ( $L =$  being the size of the container), *i.e.*, the momentum is as low as  $q_0 = \pi/L \approx 0$  and  $T \approx 0$ . Evidently, Equation 10a, representing the SS/MS states of particles, loses its significance and it can be normalized to unity. We are obviously left with Equation 10b as an effective part of the state function (Equation 10) and we find that this function is identical to the usual wave function of  $N$  plane waves representing  $N$  non-interacting particles. As such our theory gets transformed into the standard theory of non-interacting particles on setting  $V_{ij}(r) = 0$ .

It may be noted that under our valid approximations,  $V_{ij}^R(r) \approx V_{\text{HC}}(r)$  (shown to be  $\equiv A\delta(r)$ <sup>55</sup>) and  $V_{ij}^A(r) \approx -V_0$ , the solution of the Schrodinger equation of the system for its *physically possible state* (where every pair of particles satisfies  $r \geq \sigma$ ) is, obviously, a set of  $N$  plane waves. Since the SMW configuration adopted by the system is simply a consequence of the superposition of these plane waves, it is evident that our theory is consistent with the translational invariance of the fluid. We find that  $V_{\text{HC}}(r) \equiv \delta(r)$ -repulsion poses no problem of divergence of energy expectation<sup>55</sup>.

The formation of a SMW from the superposition of two plane waves of two HC particles is as natural as the phenomena of interference and diffraction involving strongly interacting particles such as electrons, neutrons,  $\text{He}$  atoms, *etc.*<sup>84</sup>. Since the nature of interference and diffraction patterns for these strongly interacting particles does not differ from the nature of such patterns for non-in-

teracting photons, it is evident that only wave nature (not the inter-particle interactions) decides the  $\phi$  positions of particles (interacting or non-interacting) in their wave mechanical superposition, which means that the formation of a SMW of two interacting particles such as  ${}^4\text{He}$  is supported by all these experiments.

Although, as concluded in Section 5.4, two particles in  $(\mathbf{q}, -\mathbf{q})$  pair configuration do form a bound pair at  $T < T_\lambda$ , this binding culminates into a collective binding among all the  $N$  atoms and leads to the formation of a single macromolecule of  $N$  atoms where no two atoms can be identified to represent  $\text{He}_2$  type molecular unit. In this context we also note that this binding binds particles in  $\phi$ -,  $q$ -, as well as  $r$ -space, and, for this reason, it can not be equated with an interatomic binding *purely* in  $r$ -space of diatomic molecule like  $\text{O}_2$ . As discussed in our third paper<sup>56</sup>, atoms in a Fermi system too can have collective binding but the fact that only two particles in such a system can have identical  $K$  may help in distinguishing one pair of fermions from the other and one may identify each pair as a diatomic molecule.

Since this theory has been successfully developed within the framework of the *wave mechanics* (in variance with a speculative remark of Putterman<sup>5</sup>, *cf.* Section 1.0), it is evident that the wave mechanics is well equipped with necessary principles for explaining the superfluidity of *He-II*. As discussed in Section 5.3, our theory also answers Putterman's question<sup>5, p. XXI</sup>, "how in a single component system there can exist two fluids with independent velocities one of which has a quantized circulation." It also proves that microscopic theory of a simple many body quantum system like liquid  ${}^4\text{He}$  needs not be as complicated as its CTs.

Finally, we find<sup>56</sup> that the framework of our theory has great potential for unifying the physics of widely different many body quantum systems of bosons as well as fermions. It is possible that the method of second quantization may have some advantages over the usual method of solving the Schrodinger equation but this study establishes that the latter is an equally versatile approach to reveal the physics of a many body quantum system like liquid helium. In fact the A2 group of CTs (*cf.* Section 1.0) based on  $g(r)$  and  $S(Q)$  do use the latter methodology but not the way we have used; they emphasize  $g(r)$  rather than  $g(\phi)$ , although  $\phi$ -position of a particle is more relevant than its  $r$ -position when it behaves like a wave. Further since two particles in their wave mechanical superposition leading to  $g(\phi)$  are the nearest neighbor with  $r = d = \lambda/2$ , their effective interaction is  $V_{\text{HC}}(r) \equiv A\tilde{\alpha}(r)$  leading to zero point repulsion. Evidently, *hypernetted chain* (HNC) Schrodinger equation used for finding  $g(r)$ <sup>38,40</sup> loses its relevance for finding  $g(\phi)$ . In addition our theory helps in understanding the system in terms of both (i)  $F$  expressed as a function of a suitable OP (*cf.* Appendix A), as well as (ii)  $\phi$ -corelations, G-state properties and  $K = 0$  condensate fraction,  $n_{K=0}(T^*)$  (*cf.* Sections 2.0–5.0). A significant amount of work related to a detailed study of excitation spectrum and quantum vortices in a SIB has been completed by the author and will be submitted soon for publication.

## APPENDIX A

### Free Energy and Order Parameter

We find that (i) each particle in the system represents a  $(\mathbf{q}, -\mathbf{q})$  pair moving with CM momentum  $\mathbf{K}$  and total energy  $E = \varepsilon_0 + \hbar^2 K^2 / 8m$  (cf. Equation 47b) of its  $q$  ( $= q_0 = \pi/d$ ) and  $K$  motions, and (ii) to a good approximation (valid even at  $T > T_\lambda$  at least near the  $\lambda$ -point), each of the  $N - N^*(T)$  particles (with  $N^*(T)$  representing particles in the excited state of  $K \geq 2\pi/\sigma$ ) exhibits quantum correlation (*equivalent potential* of  $U_{ij}^s = -k_B T_0 \cdot \ln 2$ ) with its neighbors;  $N^*(T)$  particles lack quantum correlation for their excitation wave length  $\Lambda$  being  $< \sigma$ . Consequently, the free energy  $F$  of the system can be expressed as

$$F = F(q) + F(K), \quad (\text{A-1})$$

with

$$F(q) = N\varepsilon_0 - [N - N^*(T)]k_B T_0 \cdot \ln 2 \quad (\text{A-2})$$

and

$$\begin{aligned} F(K) &= k_B T \frac{2\pi(8mk_B T)^{3/2}}{h^3} \int_0^\infty x^{1/2} \ln(1 - ze^{-x}) dx \\ &= k_B T \frac{1}{\lambda^3} g_{5/2}(z). \end{aligned} \quad (\text{A-3})$$

While  $U_{ij}^s$ , as such is a *fictitious potential*, it manifests itself as a real interaction at  $T \leq T_\lambda$  at which  $q$  and  $r$  have interdependence through  $2qr = 2n\pi$  and particles get locked at relative phase positions  $\Delta\phi = 2n\pi$ . This is well evident from the existence of the well known zero-point repulsion which does not allow WPs of two particles to share common  $r$ . Since the effect does not exist when  $\lambda/2 < d$  (*i.e.*, for  $T > T_\lambda$ ), and the net effect of the quantum correlations at  $T > T_\lambda$  represented by  $U(T > T_\lambda) = -[N - N^*(T)]k_B T_0 \ln 2$  serves as the inoperative part of such potential,  $U(T_\lambda) = [N - N^*(T_\lambda)] k_B T_0 \ln 2$  can be used as its zero level. Using these facts, we have

$$F = N\varepsilon_0 + F(K), \quad \text{for } T > T_\lambda \quad (\text{A-4})$$

and

$$F = N\varepsilon_0 - [N - N^*(T)]k_B T_0 \cdot \ln 2 + F(K), \quad \text{for } T > T_\lambda. \quad (\text{A-5})$$

Since  $K$  motions define the thermal excitations of the system both in S- and N-phases, and superfluidity is observed at all  $T \leq T_\lambda$  including  $T = 0$ , at which thermal excitations cease to exist,  $F(K)$  is not expected to play any key role in accounting for the phenomenon. Naturally,  $F(q)$  must explain superfluidity.

As usual, we may express  $F$  as a function of some suitable OP. To identify

the right OP we note that all superfluid related properties, *viz.* (i) the onset of an order in the positions of particles in  $r$ -,  $q$ - and  $\phi$ -spaces, (ii) collective nature of thermal excitations for  $Q < 2\pi/\sigma$ , (iii) negative volume expansion coefficient, (iv) increased quantum correlation among the particles, (v) coherence and laser-like behavior, (vi) increasing strength of collective binding, (vii) logarithmic singularity of specific heat and related properties, *etc.*, as concluded by our theory and supported by experiments on *He-II* cooled through  $T_\lambda$ , are the consequence of WP manifestation of particles. These are also related to the seemingly interrelated aspects, *viz.* (i) perturbative effect of  $V_{ij}^\Delta(r)$ , (ii) inter-particle quantum correlations, (iii)  $K = 0$  condensate (expressed by  $n_{K=0} = N_{K=0}/N$ , the fraction of particles condensed into the G-state, *i.e.*, a state of  $K = 0$  and  $q = \pi/d$ ), whose magnitude increases with decreasing  $T$  below  $T_\lambda$ . Obviously, the most suitable OP should be none other than  $n_{K=0}$ . We, therefore, have

$$F(T, n_{K=0}) = F_0 + \frac{1}{2}A(n_{K=0})^2 + \frac{1}{4}B(n_{K=0})^4 + \frac{1}{6}C(n_{K=0})^6 + \dots, \quad (\text{A-6})$$

where  $F_0$  (independent of  $n_{K=0}$ ) and  $A, B, C, \dots$  are smooth function of  $T$  and similar physical variables of the system. Defining  $A = \alpha(T - T_\lambda)/T_\lambda$ , we find that

$$n_{K=0}(T) \propto \sqrt{\frac{T_\lambda - T}{T_\lambda}}. \quad (\text{A-7})$$

Renormalising  $n_{K=0}(T)$  through  $T^* = T - T_0$ , we have

$$n_{K=0}(T^*) \propto \sqrt{\frac{T_\lambda^* - T^*}{T_\lambda^*}} = \sqrt{\frac{T_\lambda - T}{T_\lambda - T_0}}. \quad (\text{A-8})$$

As shown in Figure 3, we find that the experimental  $\rho_s(T)$  (a well known OP used in the phenomenological theory of Landau) for *He-II* (*cf.* Curves E1 and E2) does not match with Equation A-7 (Curve-C). Instead it matches closely with  $n_{K=0}(T^*)$  (Equation A-8, Curve-C\*) and  $n_{K=0}(T^*)$  (Equation 42b, Curve-B\*). This not only proves the consistency of Equation A-8 and different relations for  $n_{K=0}(T^*)$ , but also establishes the accuracy of the present theory. We also find that out of the  $N^*(T_\lambda)$  particles lacking quantum correlation at  $T_\lambda$ ,  $[N^*(T_\lambda) - N^*(T)]$  assume the correlations on reaching at  $T < T_\lambda$  from  $T_\lambda$ . Defining

$$n^*(T) = [N^*(T_\lambda) - N^*(T)]/N^*(T_\lambda) \quad (\text{A-9})$$

we find that experimental  $\rho_s(T)$  of *He-II* (Curves E1 and E2) shows better agreement with  $n^*(T)$  (Curve-A) as a function of  $T$  (not of  $T^*$ ). It appears that due to the effect of HC interaction  $n_{K=0}(T)$  in a SIB like LHE-4 is better represented by  $n^*(T)$ . In summary, the close agreement of experimental  $\rho_s(T)$  with (i)  $n_{K=0}(T^*)$  (Equation 42b and Curve-B\*), (ii)  $n_{K=0}(T^*)$  (Equation A-8 and

Curve-C\*) and (iii)  $n^*(T)$  (Equation A-9 and Curve-A) (cf. Figure 3 for *He-II*), and a relation of  $n_{k=0}(T^*)$  and  $n^*(T)$  (as defined in this study) with the wave nature of particles, imply that  $[N^*(T_\lambda) - N^*(T)]k_B T_0 \cdot \ln 2 = \Delta \mathcal{E}(T) = E_g(T)$  (Equations 33 and 34) is the most relevant part of  $F$  and  $n^*(T)$  or  $n_{k=0}(T^*)$  should be the most suitable OP of the  $\lambda$ -transition.

### Notes

1. Wilks J. (1967). *The Properties of Liquid and Solid Helium*. Oxford: Clarendon Press.
2. Feenberg, E. (1969). *Theory of Quantum Fluids*. New York: Academic Press.
3. Galasiewics, Z. M. (1971). *Helium 4*. Oxford: Pergamon.
4. Woods, A.D.B., & Cowley, R. A. (1973). *Reports on the Progress of Physics*, 36, 1135.
5. Putterman, S. J. (1974). *Superfluid Hydrodynamics*. Amsterdam: North-Holland.
6. Armitage, J.G.M., & Ferquhar, I. E. (Eds.) (1975). *The Helium Liquids*. London: Academic Press.
7. Benneman, K. H., & Ketterson, J. B. (Eds.) (1976). *The Physics of Liquid and Solid Helium (Part-I)*. New York: Wiley.
8. Tilley, D. R., & Tilley, J. (1986). *Superfluidity and Superconductivity*. Bristol: Adam Hilger Ltd.
9. Glyde, H. R., & Svenson, E. C. (1987). In Rice, D. L., & Skold K. (Eds.), *Methods of Experimental Physics* (Vol. 23, Part B) (pp. 303–403). San Diego: Academic.
10. Griffin, A., Snoke, D. W., & Stringari, A. (Eds.) (1995). *Bose Einstein Condensation*. Cambridge: Cambridge University Press.
11. (a) Allen, J. F., & Misener, A. P. (1938). *Nature* 141, 5. (b) Kapitza, P. (1938). *Nature* 141, 74.
12. London, F. (1938). *Physical Review*, 54, 947.
13. (a) Bose, S. N. (1924). *Zeif Physics*, 26, 178. (b) Einstein, A. (1924 & 1925). *Sitzber. Preuss Acad. Wiss.*, (1924), 261; (1925), 3.
14. Landau, L. D. (1941 & 1947). *Journal of Physics (USSR)*, (1941), 5, 71; and (1947), 11, 91. Reprinted in English in reference 3, pp. 191–233, pp. 243–246.
15. Bogoliubov, N. N. (1947). *Journal of Physics (USSR)*, 11, 23; Reprinted in English in reference 3, pp. 247–267.
16. Huang, K., & Yang, C. N. (1957). *Physical Review*, 105, 767.
17. Jastrow, R. (1955). *Physical Review*, 98, 1479.
18. Woo, C. W. in reference 7, pp. 349–501.
19. Moroni, S., Senatore, G., & Fantoni, S. (1997). *Physical Review*, B55, 1040.
20. Sokol, P. E. in reference 10, pp. 51–85.
21. Beliaev, S. T. (1958). *Soviet Physics JETP*, 7, 289.
22. Beliaev, S. T. (1958). *Soviet Physics JETP*, 7, 299.
23. Hugenholtz, N. M., & Pines, D. (1959). *Physical Review*, 116, 489.
24. Lee, T. D., & Yang, C. N. (1957). *Physical Review*, 105, 1119.
25. Brueckner, K. A., & Sawada, K. (1957). *Physical Review*, 106, 1117.
26. Wu, T. T. (1959). *Physical Review*, 115, 1390.
27. Sawada, K. (1959). *Physical Review*, 116, 1344.
28. Gavoret, J., & Nozieres, P. (1959). *Annals of Physics*, 28, 349.
29. Hohenberg, P. C., & Martin, P. C. (1959). *Annals of Physics*, 34, 291.
30. DeDominicis, C., & Martin, P. (1964). *Journal of Math. Physics*, 5, 14 & 31.
31. Toyoda, T. (1982). *Annals of Physics (NY)*, 141, 154.
32. Fetter, A., & Walecka, J. (1971). *Quantum Theory of Many Particle System*. New York: McGraw-Hill.
33. Nozieres, P. in reference 10, pp. 15–30.
34. Huang, K. in reference 10, pp. 31–50.
35. Penrose, O., & Onsager, L. (1956). *Physical Review*, 104, 576.
36. Dolan, L., & Jackiw, R. (1974). *Physical Review*, D9, 3320.
37. Yang, C. N. (1962). *Review of Modern Physics*, 34, 4.
38. Croxton, C. (1975). *Introduction to Liquid State Physics*. London: John Wiley.

39. Ceperley, D. M., & Kalos, M. H. (1981). In Zabolitzky, J. (Ed.). *Recent Progress in Many-Body Theories*. New York: Springer.
40. Smith, R. A., Kallio, A., Pouskari, M., & Toropainen, P. (1979). *Nuclear Physics*, A238, 186.
41. Campbell, C. E., & Pinski, F. J. (1979). *Nuclear Physics*, A238, 210.
42. Schmidt, K. E., & Pandharipande, V. R. (1979). *Nuclear Physics*, A238, 240.
43. Reatto, L. (1979). *Nuclear Physics*, A238, 253.
44. Ristig, M. L., & Lam, P. M. (1979). *Nuclear Physics*, A238, 267.
45. Ceperley, D. M., & Kalos, M. H. (1979). In Binder, K. (Ed.). *Monte Carlo Methods in Statistical Physics*. New York: Springer.
46. Ceperley, D. M. (1991). *Journal of Statistical Physics*, 63, 1237.
47. Silver, R. N., & Sokol, P. E. (Eds.) (1989). *Momentum Distribution*. New York: Plenum Press.
48. Kallio, A., & Piilo, J. (1996). *Physical Review Letters*, 77, 4273.
49. Over the last sixty years, the subject of the quantum properties of  $N$  body SIB has been studied extensively and roughly in about a thousand research articles and 50 books have been published. It is not possible to provide a full list of these references in a research paper like this. The reader may trace these references through reference 1–10 and other references (such as references 19, 31, 39, 40, 45, 47, etc.).
50. Rickayzen, C. in reference 6, pp. 53–109.
51. Kleban, P. (1974). *Physics Letters*, 49A, 19.
52. Onsager, L. (1949). *Nuovo Cimento Suppl. No. 2*, 6, 249.
53. Feynman, R. P. (1955). In Groter, C. J. (Ed.). *Progress in Low Temperature Physics* (p. 17). Amsterdam: North-Holland.
54. (a) Jain, Y. S. *Unified Microscopic Theory of a System of Interacting Boson*, cond-mat/9712150. (b) Jain, Y. S. (1996). In *High Temperature Superconductivity* (Proceedings of the International Workshop on High Temperature Superconductivity, Rajasthan University Jaipur, Dec. 16–21, 1996) (pp. 320–327). Narosa, New Delhi (1998). (c) Paper also presented at *XX International Workshop on Condensed Matter Theories*, Pune University, Pune, Dec. 9–13, 1996.
55. Jain, Y. S. (2002). Untouched aspects of the wave mechanics of two particles in a many body quantum system. *Journal of Scientific Exploration*, 16, 67–75.
56. Jain, Y. S. (2002). Unification of the physics of interacting bosons and fermions through  $(\mathbf{q}, -\mathbf{q})$  pair correlation. *Journal of Scientific Exploration*, 16, 117–124.
57. Taubes, G. (1995). *Science*, 269, 152.
58. Lipson, S. G. (1987). *Contemporary Physics*, 28, 117.
59. Levich, B. G. (1973). *Theoretical Physics* (Vol. 4). Amsterdam: North-Holland.
60. Uhlenbeck, G. E., & Gropper, L. (1932). *Physical Review*, 41, 79.
61. Jain, Y. S. (Unpublished work).
62. Chowdhury, D. R. (1996). Study of the Dynamical Behaviour of Certain Many Body Systems. Ph.D. thesis, Department of Physics, North-Eastern Hill University.
63. Feynman, R. P. (1976). In *Statistical Mechanics* (p. 284). Reading, MA: W. A. Benjamin.
64. Feynman, R. P., & Cohen, M. (1956). *Physical Review*, 102, 1189.
65. Karpus, M., & Porter, R. N. (1970). *Atoms and Molecules*. Menlo Park: The Benjamin/Cummings.
66. This is evident from the fact that collective effect of binding energy per particle,  $-E_b(T)/N$  (being  $\ll k_B T_0$ ) due to phase correlation among particles of  $\xi(T)^3$  volume ( $\xi(T)$  = coherence length. Equation 38) turns out to be  $E_b = -(\xi(T)^3/d^3)(E_g(T)/N) \approx -4\pi k_B T_0(\xi/d)$ , showing  $|E_b| \gg k_B T_0$ ; note that  $\xi$  for LHE-4 is found to vary from  $\approx 10^2 d$  at  $T = 0$  to  $\infty$  at  $T_0$ .
67. Foot, C. J., & Steane, A. M. (1995). *Nature*, 376, 213.
68. Cooper, L. N. (1956). *Physical Review*, 104, 1189.
69. Goldstone, J. (1961). *Nuovo Cimento*, 19, 154.
70. Anderson, P. W. (1966). *Reviews of Modern Physics*, 38, 298.
71. Fisher, M. E. (1967). *Reports on the Progress of Physics*, 30, 615.
72. Pathria, R. K. (1976). *Statistical Mechanics*. Oxford: Pergamon Press.
73. Ahlers, G. in reference 7, Chapter 2.
74. To a good approximation, the system being a close packed arrangement of WPs could be assigned a symmetry such as hcp, fcc, etc. Naturally,  $d$  can be as large as 1.122a for fcc arrangement and as small as  $a = (V/N)^{1/3}$  for sc. In our estimates given in reference 54a, we used  $d = 1.091a$ , i.e., a bcc arrangement, but here we use  $d = a$  of sc with some estimates for all the three cases.

75. The  $\xi(T)$  (Equation 38) should be related to the spatial separation between two particles in SMW configuration keeping definite phase correlation and it appears to be the origin of: (i) nearly constant thickness ( $\approx 3 \cdot 10^{-6}$  cm) of the film observed in the beaker film flow experiment (see L.J. Campbell in reference 6, Chapter 4), (ii) the radius of vortex rings  $\approx 5 \cdot 10^{-6}$  to  $10^{-4}$  cm) observed for vortices created by moving ions (see Ref. [1], Chapter 12), etc.
76. Kittel, C. (1976). *Introduction to Solid State Physics*. New Delhi: Wiley Eastern.
77. Valatin, J. G., & Butler, D. (1958). *Nuovo Cimento*, 10, 37.
78. Girardeau, M., & Arnowitt, R. (1959). *Physical Review*, 113, 755.
79. Luban, M. (1962). *Physical Review*, 128, 965.
80. Kobe, D. (1968). *Annals of Physics*, 47, 15.
81. Brown, G. V., & Coopersmith, M. H. (1969). *Physical Review*, 178, 327.
82. Coniglio, A., Mancini, F., & Maturi, M. (1969). *Nuovo Cimento B*, 63, 227.
83. Evans, W.A.B., & Imry, Y. (1969). *Nuovo Cimento B*, 63, 155.
84. (a) Carnal, O., & Mlynek, J. (1991). *Physical Review Letters*, 66, 2689. (b) Keith, D. W., Ekstrom, C. R., Turchette, Q. A., & Prochard, D. E. (1991). *Physical Review Letters*, 66, 2693. (c) Levi, B. J. (1991), *Physics Today*, 44, 17.